Tracial states and group structure

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(Joint work with Brian Forrest and Nico Spronk)

Group C*-algebras

G – locally compact group

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There are two natural C*-completions of $L^1(G)$:

- *Reduced group C*-algebra:* $C_r^*(G) := \overline{\lambda(L^1(G))}^{\|\cdot\|} \subseteq B(L^2(G))$ where

$$\lambda(f)\xi := f * \xi; \qquad (f \in L^1(G), \xi \in L^2(G)).$$

- *Full group C*-algebra:* $C^*(G)$ is the enveloping C*-algebra of $L^1(G)$.

Group C*-algebras

Theorem (Hulanicki, 1964)

TFAE:

- G is amenable;
- $C_r^*(G) = C^*(G)$;
- The *trivial representation* $1_G : L^1(G) \to \mathbb{C}$ defined by

$$1_G(f) = \int_G f(s)\, ds$$

extends to a $*$-representation of $C_r^*(G)$.

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Theorem (Ng, 2015)

If G is separable and connected, then $C_r^*(G)$ admits a tracial state if and only if G is amenable.

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Example

If G is discrete, then

$$\ell^1(G) \ni f \mapsto \langle \lambda(f)\delta_e, \delta_e \rangle = f(e) \in \mathbb{C}$$

extends to tracial state on $C_r^*(G)$.

Classes of locally compact groups

Definition

- G is an *invariant neighbourhood group* (*IN group*) if the identity $e \in G$ admits a precompact, conjugation invariant neighbourhood;
- G is a *small invariant neighbourhood group* (*SIN group*) if the identity $e \in G$ admits a neighbourhood base of precompact, conjugation invariant neighbourhoods.

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- G is a *small invariant neighbourhood group* (*SIN group*) if the identity $e \in G$ admits a neighbourhood base of precompact, conjugation invariant neighbourhoods.

$$[\text{Abelian}] \cup [\text{discrete}] \cup [\text{compact}] \subsetneq [\text{SIN groups}] \subsetneq [\text{IN groups}]$$

Example

If G admits a precompact, conjugation invariant neighbourhood K, then

$$L^1(G) \ni f \mapsto \frac{1}{m_G(K)} \langle \lambda(f)\chi_K, \chi_K \rangle \in \mathbb{C}$$

extends to a tracial state of $C_r^*(G)$.

Tracial states of $C_r^*(G)$

Theorem (Forrest-Spronk-W., 2017)

Suppose G is compactly generated. TFAE:

1. $C_r^*(G)$ admits a tracial state;
2. G is an amenable extension of an IN group;
3. G admits an open, normal, amenable subgroup.

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Sketch.

1. $\Rightarrow$ 2.:

- Suppose τ is a tracial state of $C_r^*(G)$ and view τ as a positive definite function on G.

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- Suppose τ is a tracial state of $C_r^*(G)$ and view τ as a positive definite function on G.
- $$N_\tau := \{s \in G : \tau(s) = 1\}$$
  is a closed, normal, amenable subgroup of G.

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- $$N_\tau := \{s \in G : \tau(s) = 1\}$$
  is a closed, normal, amenable subgroup of G.
- If G is compactly generated, then G/N_τ is SIN.

□

Tracial states of $C_r^*(G)$

Theorem (Kennedy-Raum, 2017)

Suppose G is any locally compact group. $C_r^*(G)$ admits a tracial state if and only if G admits an open, normal, amenable subgroup.

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New observation: our techniques work for $C^*(G)$.

New goal: How can we relate the group structure of G with the tracial states on $C^*(G)$?

Tracial states and group structure

Definition (Forrest-Spronk-W.)

The *tracial kernel* of G is

$$N_{\mathrm{Tr}} := \{s \in G : \tau(s) = 1 \text{ for all tracial states } \tau \text{ on } C^*(G)\}.$$

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If G is a SIN group, then $N_{\mathrm{Tr}} = \{e\}$.

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Example

If G is a SIN group, then $N_{\mathrm{Tr}} = \{e\}$.

Theorem (Forrest-Spronk-W.)

Suppose G is compactly generated. N_{Tr} is the smallest closed, normal subgroup N of G such that G/N is SIN.

Corollary

If G is compactly generated, then $N_{\mathrm{Tr}} = \{e\}$ if and only if G is SIN.

Tracial states and group structure

Example

Consider $G = \mathbb{R} \rtimes \mathbb{R}^{>0}$. Then

$$N_{\mathsf{Tr}} = \mathbb{R} \times \{1\}.$$

Tracial states and group structure

Definition

G is *maximally almost periodic* if for every $s \in G\backslash\{e\}$, there exists a finite dimensional representation π of G such that $\pi(s) \neq 1$.

Example

If G is maximally almost periodic, then $N_{\mathrm{Tr}} = \{e\}$ since the positive definite functions

$$\mathrm{tr} \circ \pi$$

corresponds to a tracial states of $C^*(G)$ for every finite dimensional representation of G.

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corresponds to a tracial states of $C^*(G)$ for every finite dimensional representation of G.

- Recover the classical result that every compactly generated maximally almost periodic group is SIN.

Tracial states and group structure

$$[\text{MAP groups}] \not\subseteq [\text{SIN groups}] \quad \text{and} \quad [\text{SIN groups}] \not\subseteq [\text{MAP groups}]$$

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- There exists separable, amenable $G \notin [\text{SIN groups}]$ such that
$$N_{\mathrm{Tr}} = \{e\}.$$

Amenable tracial states and group structure

Definition

Let A be a C*-algebra. A tracial state $\tau \in A^*$ is an *amenable trace* if the map

$$\begin{aligned} A \odot A^{\text{op}} &\to \mathbb{C} \\ a \otimes b^{\text{op}} &\mapsto \tau(ab) \end{aligned}$$

is $\|\cdot\|_{\text{min}}$-continuous.

- If A is nuclear, then every tracial state of A is an amenable trace.

Theorem (Brown, 2006; Forrest-Spronk-W., 2017)

G is amenable if and only if $C_r^*(G)$ admits an amenable trace.

Amenable tracial states and group strucutre

Definition (Forrest-Spronk-W.)

The *amenable tracial kernel* of G is

$$N_{\mathrm{amTr}} := \{s \in G : \tau(s) = 1 \text{ for all amenable traces } \tau \text{ on } C^*(G)\}.$$

- If $C^*(G)$ is nuclear (e.g. G connected), then $N_{\mathrm{amTr}} = N_{\mathrm{Tr}}$.

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- If $C^*(G)$ is nuclear (e.g. G connected), then $N_{\mathrm{amTr}} = N_{\mathrm{Tr}}$.

Example

If G is maximally almost periodic, then $N_{\mathrm{amTr}} = \{e\}$.

Amenable tracial states and group strucutre

Theorem (Forrest-Spronk-W.)

If G is discrete and has property (T), then $N_{\mathrm{amTr}} = \{e\}$ if and only if G is maximally almost periodic.

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Theorem (Forrest-Spronk-W.)

If G is discrete and has property (T), then $N_{\mathrm{amTr}} = \{e\}$ if and only if G is maximally almost periodic.

Corollary (Forrest-Spronk-W.)

There exists an infinite, discrete G with property (T) so that $N_{\mathrm{amTr}} = G$.

Thank you!