

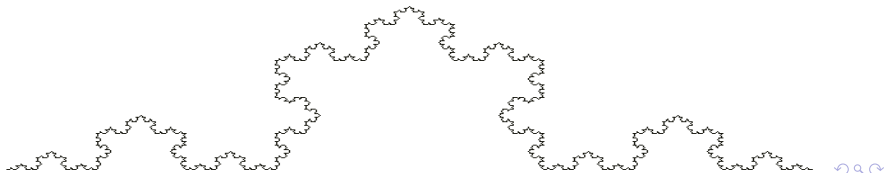
Self-similar groups and their generalisations

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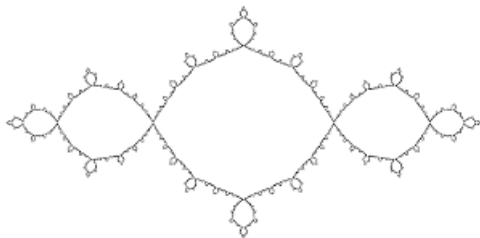
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Plan

1. Self-similar groups
2. Self-similar groupoids
3. C^* -algebras of self-similar groupoids
4. KMS states on self-similar groupoids

1. Self-similar groups



- R. Grigorchuk, *On the Burnside problem on periodic groups*, Funkts. Anal. Prilozen. **14** (1980), 53–54.
- R. Grigorchuk, *Milnor Problem on group growth and theory of invariant means*, Abstracts of the ICM, 1982.
- V. Nekrashevych, *Self-Similar Groups*, Math. Surveys and Monographs, vol. 117, Amer. Math. Soc., Providence, 2005.

Self-similar groups

- Suppose X is a finite set of cardinality $|X|$;
 - let X^n denote the set of words of length n in X with $X^0 = \emptyset$,
 - let $X^* = \bigcup_{n \geq 0} X^n$.

Definition

Suppose G is a group acting faithfully on X^* . We say (G, X) is a *self-similar group* if, for all $g \in G$ and $x \in X$, there exist $h \in G$ such that

$$g \cdot (xw) = (g \cdot x)(h \cdot w) \quad \text{for all finite words } w \in X^*. \quad (1)$$

Faithfulness of the action implies the group element h is uniquely defined by $g \in G$ and $x \in X$. So we define $g|_x := h$ and call it *the restriction of g to x* .

Then (1) becomes

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w) \quad \text{for all finite words } w \in X^*.$$

Self-similar groups

We may replace the letter x by an initial word $v \in X^k$:

For $g \in G$ and $v \in X^k$, define $g|_v \in G$ by

$$g|_v = (g|_{v_1})|_{v_2} \cdots |_{v_k}.$$

Then the self-similar relation becomes

$$g \cdot (vw) = (g \cdot v)(g|_v \cdot w) \quad \text{for all } w \in X^*.$$

Lemma

Suppose (G, X) is a self-similar group. Restrictions satisfy

$$g|_{vw} = (g|_v)|_w, \quad gh|_v = g|_{h \cdot v} h|_v, \quad g|_v^{-1} = g^{-1}|_{g \cdot v}$$

for all $g, h \in G$ and $v, w \in X^$.*

Example: the odometer

- Suppose $X = \{0, 1\}$ and $\text{Aut } X^*$ is the automorphism group.
- Define an automorphism in $\text{Aut } X^*$ recursively by

$$a \cdot 0w = 1w \qquad a \cdot 1w = 0(a \cdot w)$$

for every finite word $w \in X^*$

- The self-similar group generated by a is the integers $\mathbb{Z} := \{a^n : n \in \mathbb{Z}\}$, and (\mathbb{Z}, X) is commonly called the odometer because the self-similar action is “adding one with carryover, in binary.”

Example: the Grigorchuk group

- Suppose $X = \{x, y\}$ and $\text{Aut } X^*$ is the automorphism group.
- The Grigorchuk group is generated by four automorphisms $a, b, c, d \in \text{Aut } X^*$ defined recursively by

$$a \cdot xw = yw$$

$$b \cdot xw = x(a \cdot w)$$

$$c \cdot xw = x(a \cdot w)$$

$$d \cdot xw = xw$$

$$a \cdot yw = xw$$

$$b \cdot yw = y(c \cdot w)$$

$$c \cdot yw = y(d \cdot w)$$

$$d \cdot yw = y(b \cdot w).$$

Proposition

The generators a, b, c, d of G all have order two, and satisfy $cd = b = dc$, $db = c = bd$ and $bc = d = cb$. The self-similar action (G, X) is contracting with nucleus $\mathcal{N} = \{e, a, b, c, d\}$.

Properties of the Grigorchuk group

Theorem (Grigorchuk 1980)

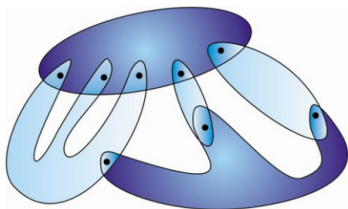
The Grigorchuk group is a finitely generated infinite 2-torsion group.

Theorem (Grigorchuk 1984)

The Grigorchuk group has intermediate growth.

(Solved a Milnor problem from 1968)

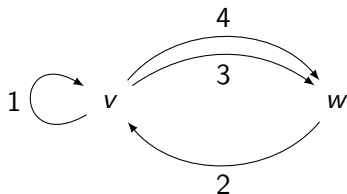
2. Self-similar groupoids



- E. Bédos, S. Kaliszewski and J. Quigg, *On Exel-Pardo algebras*, *J. Operator Theory* **78** (2017) 309–346.
- R. Exel and E. Pardo, *Self-similar graphs: a unified treatment of Katsura and Nekrashevych C^* -algebras*, *Advances in Math.* **306** (2017) 1046–1129.
- M. Laca, I. Raeburn, J. Ramagge, and M. Whittaker *Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs*, *Advances in Math.* **331** (2018) 268–325.

Directed graphs

- Let $E = (E^0, E^1, r, s)$ be a finite directed graph with vertex set E^0 , edge set E^1 , and range and source maps from E^1 to E^0 .



- Given a graph E , the set of paths of length k is

$$E^k := \{\mu = \mu_1\mu_2 \cdots \mu_k : \mu_i \in E^1, s(\mu_i) = r(\mu_{i+1})\},$$

and let

$$E^* = \bigcup_{k=0}^{\infty} E^k$$

denote the collection of finite paths. A path of length zero is defined to be a vertex.

Partial isomorphisms on graphs

- Suppose $E = (E^0, E^1, r, s)$ is a directed graph. A *partial isomorphism* of the path space E^* consists of two vertices $v, w \in E^0$ and a bijection $g : vE^* \rightarrow wE^*$ such that
 - $g(vE^k) = wE^k$ for all $k \in \mathbb{N}$ and
 - $g(\mu\nu) \in g(\mu)E^*$ for all $\mu\nu \in E^*$.
- For each $v \in E^0$ we let $\text{id}_v : vE^* \rightarrow vE^*$ denote the partial isomorphism $\text{id}_v(\mu) = \mu$ for all $\mu \in vE^*$.
- We write g for the triple $(g, s(g) := v, r(g) := w)$, and we denote the set of all partial isomorphisms on E by $\mathcal{P}(E^*)$.

Groupoids

- A *groupoid* G with unit space X consists of
 - a set G and a subset $X \subseteq G$,
 - maps $r, s : G \rightarrow X$,
 - a set $G^{(2)} = G \times_r G := \{(g, h) \in G \times G : s(g) = r(h)\}$ together with a partially defined product $(g, h) \in G^{(2)} \mapsto gh \in G$, and
 - an inverse operation $g \in G \mapsto g^{-1} \in G$with some properties.

Proposition

Suppose E is a directed graph. The set $\mathcal{P}(E^*)$ of partial isomorphisms on E^* is a groupoid with unit space E^0 . For $g : vE^* \rightarrow wE^*$ in $\mathcal{P}(E^*)$ we define

- $r(g) = w$ and $s(g) = v$,
- if $s(g) = r(h)$, the product $gh : s(h)E^* \rightarrow r(g)E^*$ is composition, and
- $g^{-1} : r(g)E^* \rightarrow s(g)E^*$ is the inverse of g .

Groupoid actions

Suppose that E is a directed graph and G is a groupoid with unit space E^0 .

- An *action* of G on the path space E^* is a (unit-preserving) groupoid homomorphism $\phi : G \rightarrow \mathcal{P}(E^*)$.
- The action is *faithful* if ϕ is injective.
- If the homomorphism is fixed, we usually write $g \cdot \mu$ for $\phi_g(\mu)$.
 - This applies in particular when G arises as a subgroupoid of $\mathcal{P}(E^*)$, which is how we will define examples.

Self-similar groupoids

Definition

Suppose E is a directed graph and G is a groupoid with unit space E^0 acting faithfully on E^* . Then (G, E) is a *self-similar groupoid* if, for every $g \in G$ and $e \in s(g)E^1$, there exists $h \in G$ satisfying

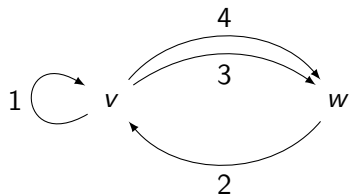
$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu) \quad \text{for all } \mu \in s(e)E^*. \quad (2)$$

- Since the action is faithful, there is then exactly one such $h \in G$, and we write $g|_e := h$.
- Now, for $g \in G$ and $\mu \in s(g)E^*$, the analogous definitions to the self-similar group case give us the formula:

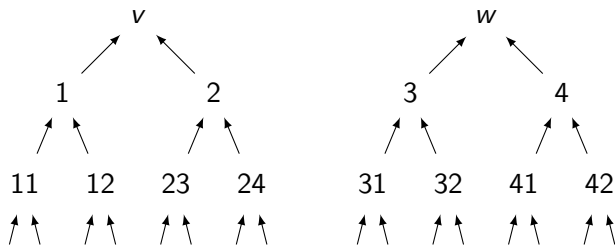
$$g \cdot (\mu\nu) = (g \cdot \mu)(g|_\mu \cdot \nu) \quad \text{for all } \nu \in s(\mu)E^*.$$

Example 1

- Let E be the graph

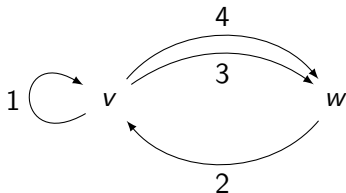


- The path space E^* is



Example 1

- Let E be the graph



- Define partial isomorphisms $a, b \in \mathcal{P}(E^*)$ recursively by

$$\begin{aligned} a \cdot 1\mu &= 4\mu & b \cdot 3\mu &= 1\mu & (3) \\ a \cdot 2\nu &= 3(b \cdot \nu) & b \cdot 4\mu &= 2(a \cdot \mu). \end{aligned}$$

- Let G be the subgroupoid of $\mathcal{P}(E^*)$ generated by A . Then (G, E) is a self-similar groupoid.

3. C^* -algebras of self-similar groupoids



- R. Exel and E. Pardo, *Self-similar graphs: a unified treatment of Katsura and Nekrashevych C^* -algebras*, *Advances in Math.* **306** (2017) 1046–1129.
- M. Laca, I. Raeburn, J. Ramagge, and M. Whittaker *Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs*, *Advances in Math.* **331** (2018) 268–325.
- V. Nekrashevych, *C^* -algebras and self-similar groups*, *J. Reine Angew. Math.* **630** (2009), 59–123.

C^* -algebras of self-similar groupoids

Proposition

Let E be a finite graph without sources and (G, E) a self-similar groupoid action. There is a Toeplitz algebra $\mathcal{T}(G, E)$ defined by families $\{p_v : v \in E^0\}$, $\{s_e : e \in E^1\}$ and $\{u_g : g \in G\}$ such that

1. u is a unitary representation of G with $u_v = p_v$ for $v \in E^0$;
2. (p, s) is a Toeplitz-Cuntz-Krieger family in $\mathcal{T}(G, E)$, and $\sum_{v \in E^0} p_v$ is an identity for $\mathcal{T}(M)$;
3. if $g \in G$ and $e \in E^1$ with $s(g) = r(e)$, then

$$u_g s_e = s_{g \cdot e} u_{g|_e}$$

4. if $g \in G$ and $v \in E^0$ with $s(g) = v$, then

$$u_g p_v = p_{g \cdot v} u_g.$$

C^* -algebras of self-similar groupoids

Proposition

Let (p, s, u) be the universal representation of the Toeplitz algebra $\mathcal{T}(G, E)$. Then

$$\mathcal{T}(G, E) = \overline{\text{span}}\{s_\mu u_g s_\nu^* : \mu, \nu \in E^*, g \in G \text{ and } s(\mu) = g \cdot s(\nu)\}.$$

Proposition

Let (p, s, u) be the universal representation of the Toeplitz algebra $\mathcal{T}(G, E)$. Then the Cuntz-Pimsner algebra $\mathcal{O}(G, E)$ is the quotient of $\mathcal{T}(G, E)$ by the ideal generated by

$$\left\{ p_\nu - \sum_{\{e \in \nu E^1\}} s_e s_e^* : \nu \in E^0 \right\}.$$

The gauge action

- There are natural \mathbb{R} -automorphic dynamics on $\mathcal{T}(G, E)$ and $\mathcal{O}(G, E)$ defined by

$$\sigma_t(p_v) = p_v, \quad \sigma_t(s_e) = e^{it} s_e \quad \text{and} \quad \sigma_t(u_g) = u_g$$

- We are interested in (KMS) equilibrium states of the dynamical systems $(\mathcal{T}(G, X), \sigma)$ and of $(\mathcal{O}(G, X), \sigma)$.

4. KMS states on self-similar groupoids



- M. Laca, I. Raeburn, J. Ramagge, and M. Whittaker *Equilibrium states on the Cuntz-Pimsner algebras of self-similar actions*, J. Func. Anal. **266** (2014), 6619–6661.
- M. Laca, I. Raeburn, J. Ramagge, and M. Whittaker *Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs*, Advances in Math. **331** (2018) 268–325.
- J. Claramunt and A. Sims *Preferred traces on C^* -algebras of self-similar groupoids arising as fixed points*, arXiv:1712.00302.

The KMS condition

- Suppose $\sigma : \mathbb{R} \rightarrow \text{Aut}(A)$ is a strongly continuous action, then there is a dense $*$ -subalgebra of σ -analytic elements:
 $t \mapsto \sigma_t(a)$ extends to an entire function $z \mapsto \sigma_z(a)$.

The KMS condition

- Suppose $\sigma : \mathbb{R} \rightarrow \text{Aut}(A)$ is a strongly continuous action, then there is a dense $*$ -subalgebra of σ -analytic elements: $t \mapsto \sigma_t(a)$ extends to an entire function $z \mapsto \sigma_z(a)$.

- **Definition**

The state φ of A satisfies the KMS condition at inverse temperature $\beta \in (0, \infty)$ if

$$\varphi(ab) = \varphi(b \sigma_{i\beta}(a))$$

whenever a and b are analytic for σ .

KMS states on the Toeplitz algebra

Theorem

Suppose E is a strongly connected finite graph with no sources. Let B be the vertex matrix of E with spectral radius $\rho(B)$.

- 1. If $\beta \in [0, \log \rho(B))$, there are no KMS_β states on $(\mathcal{T}(G, E), \sigma)$;*
- 2. If $\beta \in (\log \rho(B), \infty)$, there is a homeomorphism between the normalised traces on the groupoid C^* -algebra $C^*(G)$ and the KMS_β states on $(\mathcal{T}(G, E), \sigma)$;*
- 3. If $\beta = \log \rho(B)$, the $\text{KMS}_{\log \rho(B)}$ states of $(\mathcal{T}(G, E), \sigma)$ arise from KMS states of $(\mathcal{O}(G, E), \sigma)$; and there is at least one such state.*
- 4. If the set $\{g|_\mu : \mu \in E^*\}$ is finite for every $g \in G$, then this is the only KMS state of $(\mathcal{O}(G, E), \sigma)$.*

The unique KMS state

- Suppose that E is a finite graph with no sources, that E is strongly connected, and that (G, E) is a self-similar groupoid action such that the set $\{g|_{\mu} : \mu \in E^*\}$ is finite for every $g \in G$.
- In this situation the vertex matrix B is irreducible, and has a unique unimodular Perron-Frobenius eigenvector $x \in (0, \infty)^{E^0}$.
- For $g \in G$, $v \in E^0$ and $k \geq 0$, define

$$F_g^k(v) := \{\mu \in s(g)E^k v : g \cdot \mu = \mu \text{ and } g|_{\mu} = \text{id}_v\}, \text{ and}$$
$$c_{g,k} := \rho(B)^{-k} \sum_{v \in E^0} |F_g^k(v)| x_v.$$

- Then for each $g \in G \setminus E^0$, the sequence $\{c_{g,k} : k \in \mathbb{N}\}$ is increasing and converges with limit c_g in $[0, x_{s(g)}]$.

The unique KMS state

Theorem

In the situation from the last slide, the unique $KMS_{\log \rho(B)}$ state of $(\mathcal{O}(G, E), \sigma)$ is given by

$$\psi(s_\kappa u_g s_\lambda^*) = \begin{cases} \rho(B)^{-|\kappa|} c_g & \text{if } \kappa = \lambda \text{ and } s(g) = r(g) = s(\kappa) \\ 0 & \text{otherwise.} \end{cases}$$

- So we need to compute the values of c_g . This is achieved by evaluating the limit

$$c_g = \lim_{k \rightarrow \infty} \rho(B)^{-k} \sum_{v \in E^0} |F_g^k(v)|_{X_v}$$

The Grigorchuk group

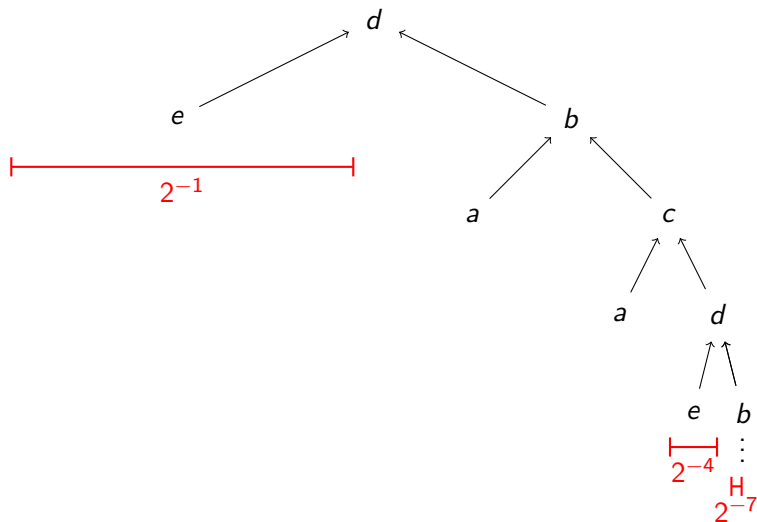
In the case of a self-similar group the graph is a bouquet of loops with a single vertex v . Thus $B = [|X|]$ has spectral radius $\rho(B) = |X|$ with unimodular Perron-Frobenius eigenvector $x_v = 1$.

Proposition

Let (G, X) be the self-similar action of the Grigorchuk group. Then $(\mathcal{O}(G, X), \sigma)$ has a unique $KMS_{\log 2}$ state ψ which is given on generators by

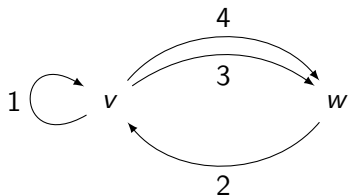
$$\psi_{\log 2}(u_g) = \begin{cases} 1 & \text{for } g = e \\ 0 & \text{for } g = a \\ 1/7 & \text{for } g = b \\ 2/7 & \text{for } g = c \\ 4/7 & \text{for } g = d. \end{cases}$$

Computation of c_d for the Grigorchuk group



$$c_d = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{3n} = \frac{1}{2} \left(\frac{1}{1 - \frac{1}{8}}\right) = \frac{4}{7}.$$

Example 1



Recall, (G, E) be the self-similar groupoid defined by:

$$\begin{aligned} a \cdot 1\mu &= 4\mu & b \cdot 3\mu &= 1\mu \\ a \cdot 2\nu &= 3(b \cdot \nu) & b \cdot 4\mu &= 2(a \cdot \mu). \end{aligned} \tag{4}$$

Proposition

The Cuntz-Pimsner algebra $(\mathcal{O}(G, E), \sigma)$ has a unique $KMS_{\log 2}$ state ψ which is given on generators by

$$\psi(u_g) = \begin{cases} 0 & \text{for } g \in \{a, b, a^{-1}, b^{-1}\}, \\ 1/2 & \text{for } g \in \{v, w\}. \end{cases}$$

A new approach to the unique KMS state on $(\mathcal{O}(G, E), \sigma)$

Claramunt and Sims recently developed a new strategy for finding the unique KMS state using self-similarity.

- Given $\beta > \log \rho(B)$ and a trace τ on $C^*(G)$, the following formula gives a KMS state $\Psi_{\beta, \tau}$ on $(\mathcal{T}(G, E), \sigma)$:

$$\Psi_{\beta, \tau}(s_\kappa u_g s_\lambda^*) = \frac{\delta_{\kappa, \lambda}}{\text{factor}} \sum_{j=|\kappa|}^{\infty} e^{-\beta j} \sum_{\{\nu \in s(\kappa) E^{j-|\kappa|} : g \cdot \nu = \nu\}} \tau(i_{g|\nu}).$$

- Restricting $\Psi_{\beta, \tau}$ to $C^*(G)$ gives back another trace on $C^*(G)$.
- Thus, there is a map $\chi_\beta : T(C^*(G)) \rightarrow T(C^*(G))$ that can be iterated.

A new approach to the unique KMS state on $(\mathcal{O}(G, E), \sigma)$

Theorem (Claramunt and Sims 2018)

Let (G, E) be a finite state self-similar groupoid where E is a finite strongly connected graph, and suppose $\beta > \log \rho(B)$. Then

- the map $\chi_\beta : T(C^*(G)) \rightarrow T(C^*(G))$ has a unique fixed point θ ;
- for any $\tau \in T(C^*(G))$ we have $\chi_\beta^n(\tau) \xrightarrow{w^*} \theta$;
- θ is the unique trace on $C^*(G)$ that extends to a $\text{KMS}_{\log \rho(B)}$ -state of $(\mathcal{O}(G, E), \sigma)$.