Positive operators on the space c_0 over a non-Archimedean valued field

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1. Positive Operators on c_0
2. Partial Order on $S(\mathcal{S}_0)$

Preliminaries

Recall from the previous talk that G is a subroup of $(\mathbb{R},+)$, K is a real closed field,

$$\mathcal{H} := K((G)) = \{f : G \to K/\text{supp}(f) \text{ is well-ordered}\},$$

and $\mathcal{H}(i) = \mathcal{H} \oplus i\mathcal{H}$.

$$c_0 = \left\{ (\lambda_j)_{j\in\mathbb{N}} : \lambda_j \in \mathcal{H}(i) \text{ for all } j \in \mathbb{N}, \lim_{j\to\infty} \lambda_j = 0 \right\}$$

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$\mathcal{A}_0$ is the space of linear continuous operators on c_0 that admit adjoints.

Recall also that $S(\mathcal{A}_0)$ is the subspace of $\mathcal{A}_0$ formed by all the operators $T \in L(c_0)$ such that the set of eigenvectors of T contains an orthonormal base of c_0; and

$S(\mathcal{S}_0)$ is the subspace of $S(\mathcal{A}_0)$ formed by all the self-adjoint operators $T \in S(\mathcal{A}_0)$.

Since K is real closed, there is an order on K that makes it into an ordered field; then we can define an order relation on $\mathcal{H}$ as follows:

For $x, y \in \mathcal{H}$, we say

$$x \leq y \text{ if } x = y \text{ or } \left(x \neq y \text{ and } (x-y)[\lambda(x-y)] < 0\right).$$

Then $(\mathcal{H}, \leq)$ is a non-Archimedean ordered field.

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Outline for section 2

1. Positive Operators on c_0

2. Partial Order on $S(\mathcal{S}_0)$

Positive Operators

<u>Definition</u>: For $T \in S(\mathcal{A}_0)$, we say that T is positive and write $T \geq 0$ if $\langle Tx, x \rangle \geq 0$ for all $x \in c_0(\mathcal{H}(i))$.

<u>Proposition</u>: Let $S, T \geq 0$ in $S(\mathcal{A}_0)$ and $\alpha \geq 0$ in $\mathcal{H}$ be given. Then

- $\alpha S + T \geq 0$.
- T is self-adjoint; that is $T \in S(\mathcal{S}_0)$.
- For all $x, y \in c_0$, we have that

$$|\langle Tx, y \rangle|^2 \leq |\langle Tx, x \rangle| \, |\langle Ty, y \rangle| \, .$$

<u>Proposition</u>: Let $T \in S(\mathcal{A}_0)$. Then both TT^* and T^*T are positive.

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Proposition: Let $T \in S(\mathcal{A}_0)$. Then both TT^* and T^*T are positive.

Theorem: For $T \in S(\mathcal{A}_0)$, the following are equivalent:

1. $T \geq 0$.
2. T is self-adjoint; and all of its eigenvalues are in $\mathcal{H}$ and non-negative.
3. There exists $S \geq 0$ in $S(\mathcal{A}_0)$ such that $T = S^2$.
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Proof: $(1) \Rightarrow (2)$: Assume $T \geq 0$. Then T is self-adjoint. Now let λ be an eigenvalue of T and let $v \in c_0(\mathcal{H}(i))$ be a corresponding eigenvector. Then

$$0 \leq \langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle.$$

Since $\langle v, v \rangle > 0$, it follows that $\lambda \in \mathcal{H}$ and $\lambda \geq 0$.

$(2) \Rightarrow (3)$: Since $T \in S(\mathcal{S}_0)$, there exist $(\lambda_n) \in \ell^\infty(\mathcal{H})$ and an orthonormal sequence $\{y_n\}$ in c_0 such that

$$T = \sum_{n=1}^{\infty} \lambda_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n.$$

Thus, λ_n is an eigenvalue of T for each n and hence $\lambda_n \in \mathcal{H}$ and $\lambda_n \geq 0$ for all $n \in \mathbb{N}$.

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Let $S : c_0 \to c_0$ be given by

$$S = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n.$$

Then $S \in S(\mathcal{A}_0)$.

For all $x \in c_0$, we have that

$$\begin{aligned} \langle Sx, x \rangle &= \sum_{n=1}^{\infty} \sqrt{\lambda_n} \frac{\langle x, y_n \rangle}{\langle y_n, y_n \rangle} \langle y_n, x \rangle \\ &= \sum_{n=1}^{\infty} \sqrt{\lambda_n} \frac{\langle x, y_n \rangle \overline{\langle x, y_n \rangle}}{\langle y_n, y_n \rangle} \geq 0. \end{aligned}$$

Hence $S \geq 0$.

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Hence $S \geq 0$.

Also, for all $x \in c_0$:

$$S^2x = S(Sx) = S\left(\sum_{n=1}^{\infty}\sqrt{\lambda_n}\frac{\langle x, y_n\rangle}{\langle y_n, y_n\rangle}y_n\right)$$

$$= \sum_{n=1}^{\infty}\sqrt{\lambda_n}\frac{\langle x, y_n\rangle}{\langle y_n, y_n\rangle}S(y_n)$$

$$= \sum_{n=1}^{\infty}\lambda_n\frac{\langle x, y_n\rangle}{\langle y_n, y_n\rangle}y_n = Tx.$$

$(3) \Rightarrow (4)$: Assume there exists $S \geq 0$ in $S(\mathcal{A}_0)$ such that $T = S^2$. Then S is self-adjoint. Thus, $S = S^*$ and hence $T = S^2 = SS = S^*S$.

$(4) \Rightarrow (1)$: Previous proposition.

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Remark: Let T and S be as in the previous theorem. Then

- S is unique. We say S is the positive square root of T and write $S = \sqrt{T}$.
- Moreover,

$$\|S\| = \|(\sqrt{\lambda_n})\| = \sup_{n\in\mathbb{N}} \left\{ |\sqrt{\lambda_n}| \right\} = \sup_{n\in\mathbb{N}} \left\{ |\lambda_n|^{1/2} \right\}$$
$$= \left[\sup_{n\in\mathbb{N}} \{|\lambda_n|\} \right]^{1/2} = \|(\lambda_n)\|^{1/2} = \|T\|^{1/2}.$$

Proposition: Let $T \geq 0$ in $S(\mathcal{A}_0)$, let $S = \sqrt{T}$, and let $R \in S(\mathcal{A}_0)$ be given. Then $TR = RT \Leftrightarrow SR = RS$.

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Corollary: Let $T \geq 0$ in $S(\mathcal{A}_0)$. Then $\langle Tx, x\rangle = 0$ for all $x \in c_0$ if and only if $T = 0$.

Proposition: Let $S, T \in S(\mathcal{A}_0)$ be positive. Then $ST \geq 0 \Leftrightarrow ST = TS$.

Proof: ($\Rightarrow$): Easy.

($\Leftarrow$): Assume $ST = TS$. Let $N = \sqrt{T}$. Applying a previous proposition, we have that $NS = SN$. Now let $x \in c_0$ be given. Then

$$\begin{aligned}\langle STx, x\rangle &= \langle S(NN)x, x\rangle = \langle (SN)Nx, x\rangle \\ &= \langle (NS)Nx, x\rangle = \langle N(SN)x, x\rangle \\ &= \langle SNx, N^*x\rangle = \langle SNx, Nx\rangle \geq 0.\end{aligned}$$

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<u>Theorem</u>: $\geq$ defines a partial order on $S(\mathcal{S}_0)$.

- Reflexivity: For all $T \in S(\mathcal{S}_0)$, $T \geq T$.
- Antisymmetry: Let $S, T \in S(\mathcal{S}_0)$ be such that $S \geq T$ and $T \geq S$. Then, for all $x \in c_0$:

  $$\langle (S-T)x, x\rangle \geq 0 \text{ and } \langle (T-S)x, x\rangle \geq 0,$$

  from which we get

  $$\langle (S-T)x, x\rangle = 0$$

  for all $x \in c_0$. It follows that $S - T = 0$ and hence $S = T$.
- Transitivity: Let $R \geq S$ and $S \geq T$. Then

  $$R - T = (R - S) + (S - T) \geq 0,$$

  and hence $R \geq T$.

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$$\langle (S-T)x, x\rangle \geq 0 \text{ and } \langle (T-S)x, x\rangle \geq 0,$$

  from which we get

$$\langle (S-T)x, x\rangle = 0$$

  for all $x \in c_0$. It follows that $S - T = 0$ and hence $S = T$.
- Transitivity: Let $R \geq S$ and $S \geq T$. Then

$$R - T = (R-S) + (S-T) \geq 0,$$

  and hence $R \geq T$.

Partial Order on $S(\mathcal{S}_0)$

Definition: For $S, T \in S(\mathcal{S}_0)$, we say that $S \geq T$ (or $T \leq S$) if $S - T \geq 0$.

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Example: Let $S, T \in S(\mathcal{S}_0)$ be given by

$$S(x) = \frac{\langle x, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle x, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2 \text{ and } T(x) = 2\frac{\langle x, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2,$$

where $(e_n)_{n \in \mathbb{N}}$ is the canonical base for c_0.

Then

$$\begin{aligned} (S-T)(x) &= \frac{\langle x, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 - \frac{\langle x, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2; \text{ and} \\ (T-S)(x) &= -\frac{\langle x, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle x, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2. \end{aligned}$$

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Results

1. If $S \geq T$ and $U \geq V$ in $S(\mathcal{S}_0)$ then $S + U \geq T + V$. Moreover, if $S \geq T$ in $S(\mathcal{S}_0)$ and $\alpha \geq 0$ in $\mathcal{H}$ then $\alpha S \geq \alpha T$.

2. For $R, S, T \in S(\mathcal{S}_0)$: $R \geq 0$ & $S \geq T \not\Rightarrow SR \geq TR$.

3. Let $S, T \in S(\mathcal{S}_0)$ be given. Then
$$S \geq T \Leftrightarrow \langle Sx, x\rangle \geq \langle Tx, x\rangle \text{ for all } x \in c_0.$$

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