NC function theory, NC convex sets, and operator theory

Lecture III: Noncommutative convexity, UCP maps and dilations

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Based on a joint works with Den Davidson, Adam Dor-On, Ben Passer and Baruch Solel

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A fresh start

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Now see whiteboard for definition of **Spectrahedron** and the **matrix cube problem**.

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Useful picture:

$$N_i = \begin{pmatrix} X_i & * \\ * & * \end{pmatrix}$$

Classical dilation theorems

Theorem (Halmos, 1950)

If $T \in B(\mathcal{H})$ is a contraction, then

$$U = \begin{pmatrix} T & \sqrt{1 - TT^*} \\ \sqrt{1 - T^*T} & -T^* \end{pmatrix}$$

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This implies that each level $\mathcal{S}_m$ is convex (use $V_i = \sqrt{t_i}\, I_n$).

Example: Free spectrahedra

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For every $A \in B(\mathcal{H})^d$, we define its free spectrahedron to be the free set

$$\mathcal{D}_A = \left\{ X \in \bigsqcup_{n=1}^{\infty} (M_n(\mathbb{C}))^d_{sa} : \sum_{j=1}^{d} X_j \otimes A_j \leq I \right\}.$$

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The set $\mathcal{D}_A(1)$ is a spectraderon. In concrete applications people usually care about $A \in B(\mathcal{H})^d$, with $\dim \mathcal{H} < \infty$. Many but not all* convex sets in $\mathbb{R}^d$ are spectrahedra with f.d. $\mathcal{H}$.

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$[-1,1]^d$ is a spectrahedron $= \mathcal{D}_C(1)$, where C as in whiteboard.

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Note $\mathcal{D}_C = d$-tuples of self-adjoint contractions.

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(An additional argument shows enough to consider $X \in \mathcal{D}_C(n)$.)

Relaxation of the matrix cube problem

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HKM observed that the problem the inclusion problem for free spectrahedra is equivalent to UCP interpolation problem, thus more tractable then that for spectrahedra (connecting to Ben-Tal and Nemirovski's earlier work in control theory).

HKMS prove that if $A \in M_n^d$ then $[-1,1]^d \subseteq \mathcal{D}_A(1) \Rightarrow \mathcal{D}_C \subseteq \vartheta_n \mathcal{D}_A$. Indeed: if $X \in \mathcal{D}_C(n)$, then by their dilation theorem $X \prec \vartheta_n N$, and $\sigma(N) \subseteq [-1,1]^d \subseteq \mathcal{D}_A(1)$. So

$$\sum X_j \otimes A_j \prec \vartheta_n \sum N_j \otimes A_j \leq I$$

(An additional argument shows enough to consider $X \in \mathcal{D}_C(n)$.)

Matricial relaxation of MCP: Test $\mathcal{D}_C \subseteq \vartheta_n \mathcal{D}_A$. If yes: then $[-1,1]^d \subseteq \vartheta_n \mathcal{D}_A(1)$. If not then $[-1,1]^d \subsetneq \mathcal{D}_A(1)$.

Example: Matrix ranges

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For every $A \in B(\mathcal{H})^d$, we define its matrix range to be the free set

$$\begin{aligned} \mathcal{W}(A) &= \cup_n \{\phi(A) : \phi \in UCP(C^*(A), M_n)\} \\ &= \cup_n \{(\phi(A_1), \ldots, \phi(A_d)) : \phi \in UCP(C^*(A), M_n)\}. \end{aligned}$$

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Matrix ranges are “dual" to free spectrahedra.

Matrix convex sets and UCP interpolation

UCP interpolation problem: Given $A \in B(\mathcal{H})^d$ and $B \in B(\mathcal{K})^d$, determine whether there exists a UCP map $OS(A) \to OS(B)$ sending A_i to B_i $(i = 1, \ldots, d)$.

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Our proof: We showed that for a normal tuple A

$$\mathcal{W}(A) = \mathcal{W}^{\min}(\operatorname{conv}(A))$$

Matrix convex sets and UCP interpolation (cont.)

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Corollary

Let $A \in B(\mathcal{H})^d$ *and* $B \in B(\mathcal{K})^d$*. There exists a unital completely isometric map* $OS(A) \to OS(B)$ *sending* A_i *to* B_i *if and only if* $\mathcal{W}(B) = \mathcal{W}(A)$*.*

Another sample result on matrix ranges

By the theorem in the previous slide, $\mathcal{W}(A)$ determines the structure of $OS(A)$ up to completely isometric isomorphism. Can the matrix range detect finer structure?

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(In passing: Minimality is closely related to the Shilov boundary).

Theorem (HKM 2013; DDSS 2016, correction* 2018)

Let A, B be two ***minimal*** *d-tuples of compact operators. Then (under a non-singularity assumption) $\mathcal{W}(A) = \mathcal{W}(B)$ if and only if A and B are unitarily equivalent.*

* — See Ben Passer's recent paper on arxiv.

A setup for a systematic study

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The set $\mathcal{S} = \bigsqcup_{n=1}^{\infty} \mathcal{S}_n$ of (self-adjoint) d-tuples is matrix convex if

1. $\mathcal{S}$ is closed under direct sums.
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$\Rightarrow$ Study the minimal and maximal matrix convex sets with ground level K.

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Violating a linear inequality is detected by a state, and matrix convex sets are closed under applications of states.

Dilations via matrix convex sets

Conclusion: for compact and convex K and L, asking whether $\mathcal{W}^{\mathrm{max}}(K) \subseteq \mathcal{W}^{\mathrm{min}}(L)$ (perhaps with L a multiple of K) is a very general matrix dilation problem.

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Suppose that $K \subseteq \mathbb{R}^d$ where K has nice symmetry or invariance properties (for example $K = -K$). Then*

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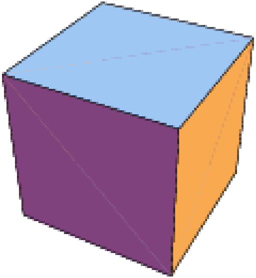
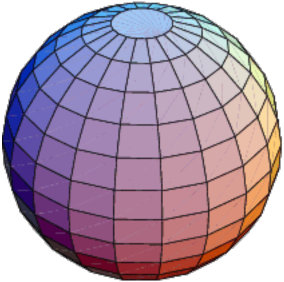
Alternatively: for $T \in B(\mathcal{H})^d$, if the joint numerical range $\mathcal{W}_1(T)$

$$\mathcal{W}_1(T) = \{\phi(T) : \phi \in UCP(B(\mathcal{H}), \mathbb{C})\} \subseteq K$$

then T has a normal dilation N with $\sigma(N) \subseteq d \cdot K$.

Examples (instead of a definition)





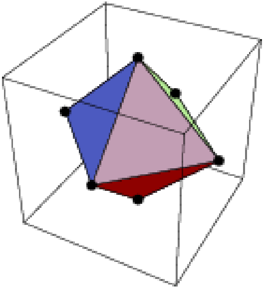
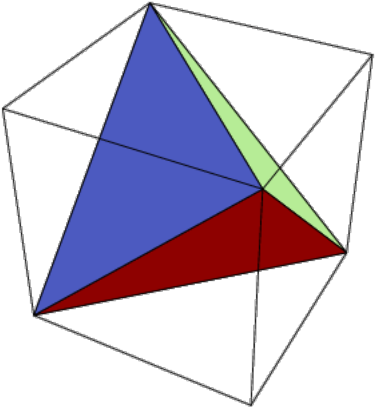


Image credit: http://mathworld.wolfram.com "Cube", "Sphere", "Octahedron"

Example - the regular tetrahedron



Does not satisfy $K = -K$.

(Also: convex hulls of frames with high symmetry, e.g., regular polytopes)

Image credit: http://mathworld.wolfram.com "Regular Tetrahedron"

The general dilation theorem (in detail)

Theorem (DDSS, 2016)

Let K, L be convex bodies in $\mathbb{R}^d$. Suppose that there exists k real $d \times d$ rank one matrices $\lambda^{(1)}, \dots, \lambda^{(k)}$ such that

1. $\lambda^{(m)} K \subseteq L$ *for all* $m = 1, \dots, k$,
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Then every X with $\mathcal{W}_1(X) \subseteq K$ has a normal dilation N with $\sigma(N) \subseteq L$.

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Thus we pre-recover [FNT 2017] result

$$\boldsymbol{K = -K \Longrightarrow \mathcal{W}^{\text{max}}(K) \subseteq d \cdot \mathcal{W}^{\text{min}}(K)}$$

Examples

$$\overline{\mathbb{B}}_d = \text{ closed unit ball of } \ell^2 \text{ space in } \mathbb{R}^d$$

$$\Delta_d = \text{ standard } d\text{-simplex: the convex hull of } 0, e_1, \ldots, e_d \in \mathbb{R}^d$$

$$D_d = d\text{-dimensional diamond: the convex hull of } \pm e_1, \ldots, \pm e_d \in \mathbb{R}^d$$

Sample results (DDSS, 2016)

$$\mathcal{W}^{\max}(\overline{\mathbb{B}}_d) \subseteq d \cdot \mathcal{W}^{\min}(\overline{\mathbb{B}}_d) \qquad \mathcal{W}^{\max}(\Delta) \subseteq d \cdot \mathcal{W}^{\min}(\Delta)$$

$$\forall C, \ \mathcal{W}^{\max}(e_1 + \overline{\mathbb{B}}_d) \not\subseteq C \cdot \mathcal{W}^{\min}(e_1 + \overline{\mathbb{B}}_d)$$

$$\mathcal{W}^{\max}([-1,1]^d) \subseteq d \cdot \mathcal{W}^{\min}(D_d) \qquad \mathcal{W}^{\max}(D_d) \subseteq 1 \cdot \mathcal{W}^{\min}([-1,1]^d)$$

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1. Computed $\theta(\cdot)$ for ℓ^p-balls and their positive sections.
2. Characterized when $K \subseteq \mathbb{R}^d$ has $\theta(K) = 1$.
3. Characterized when an ℓ^2-ball is a dilation hull for another ℓ^2-ball.
4. Computed some examples of minimal dilation hulls.

Dilation constants and the Banach-Mazur distance

Definition (“Banach-Mazur" distance)

For $K, L \subseteq \mathbb{R}^d$, define

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so $\mathcal{W}^{\max}(K) \leq \theta(L)C\mathcal{W}^{\min}(K)$, or $\theta(K) \leq C\theta(L)$.

Cube/rectangle dilation

Theorem (PSS, 2018)

For $a_1, \ldots, a_d > 0$*,*

$$\mathcal{W}^{max}([-1,1]^d) \subseteq \mathcal{W}^{min}([-a_1,a_1] \times \cdots \times [-a_d,a_d])$$

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Corollary (Using the above results, duality and "interpolation")

Let $\overline{\mathbb{B}}_{d,p}$ *denote the closed unit ball of* ℓ^p*-space in* $\mathbb{R}^d$. *Then*

$$\theta(\overline{\mathbb{B}}_{d,p}) = d^{1-|1/2-1/p|}$$

A key tool — anticommutation

Lemma

If $x_1, \ldots, x_d$ are pairwise anticommuting ($x_i x_j = -x_j x_i$), self-adjoint elements of a C^*-algebra, then

$$||x_1 + \ldots + x_d|| = \sqrt{||x_1^2 + \ldots + x_d^2||} \leq \sqrt{||x_1||^2 + \ldots + ||x_d||^2}$$

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$$||(F_1 - y_1 I) \otimes F_1 + \ldots + (F_d - y_d I) \otimes F_d|| \geq \sqrt{||y||^2 + (d-1)^2} + 1.$$

in particular

$$||F_1 \otimes F_1 + \ldots + F_d \otimes F_d|| \geq d.$$

Explicit cube dilation ($d = 2$)

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$$Y_i := \begin{pmatrix} X_i & \sqrt{1 - X_i^2} \\ \sqrt{1 - X_i^2} & -X_i \end{pmatrix}$$

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Sharpness: the anticommuting F_1, F_2 show that can't do better than $\sqrt{2}$.

Case of a rectangle (assuming $a_1^{-2} + a_2^{-2} \leq 1$)

We seek to prove: $\mathcal{W}^{\max}([-1,1]^2) \subseteq \mathcal{W}^{\min}([-a_1, a_1] \times [-a_2, a_2])$.
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$$N_1 = \begin{pmatrix} Y_1 & r \cdot \frac{1}{2}[Y_2, Y_1] \\ r \cdot \frac{1}{2}[Y_1, Y_2] & Y_1 \end{pmatrix} \qquad N_2 = \begin{pmatrix} Y_2 & \frac{1}{r} \cdot I \\ \frac{1}{r} \cdot I & -Y_2 \end{pmatrix}$$

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Suppose $K \subset \mathbb{R}^d$ is polyhedral. Then $\mathcal{W}^{\text{max}}(K) = \mathcal{W}^{\text{min}}(K)$ if and only if K is a simplex.

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Don't want to contradict that $\theta(\overline{\mathbb{B}}_d) = d$!!!

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That's not the only way dilation hulls occur:

Example

We know that $\mathcal{W}^{\text{max}}([-1,1]^2) \subset \sqrt{2} \cdot \mathcal{W}^{\text{min}}([-1,1]^2)$, but there is no triangle Π with $[-1,1]^2 \subseteq \Pi \subseteq \sqrt{2} \cdot [-1,1]^2$.

Dilating a ball to a ball

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It is easy to add in a shift and scale of the ball $\overline{\mathbb{B}}_d^2$ on the left side, too.

Thank you very much!

References

1. K.R. Davdison, A. Dor-On, O.M. Shalit and B. Solel, **“Dilations, inclusions of matrix convex sets, and completely positive maps"**, IMRN 2016.
2. B. Passer, O.M. Shalit and B. Solel, **“Minimal and maximal matrix convex sets"**, JFA 2018.

Proof of the dilation theorem

Theorem (DDSS, 2016)

Let K, L be convex bodies in $\mathbb{R}^d$. Suppose that there exists k real $d \times d$ rank one matrices $\lambda^{(1)}, \dots, \lambda^{(k)}$ such that

1. $\lambda^{(m)} K \subseteq L$ *for all* $m = 1, \dots, k$,
2. $I_d \in \mathrm{conv}\{\lambda^{(1)}, \dots, \lambda^{(k)}\}$.

Then every X with $\mathcal{W}_1(X) \subseteq K$ has a normal dilation N with $\sigma(N) \subseteq L$.

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$$S_{i,j} = \operatorname{diag}(\lambda_{i,j}^{(1)}, \ldots, \lambda_{i,j}^{(k)}). \tag{1}$$

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The construction is complete. To finish the proof one checks directly that all claims hold. $\square$

Spectrahedra

Let $A = (A_1, \dots, A_d) \in B(\mathcal{H})^d_{sa}$ (and suppose $\dim \mathcal{H} < \infty$). The spectrahedron determined by A is the set

$$\mathcal{D}_A(1) = \left\{ x \in \mathbb{R}^d : \sum_{j=1}^{d} x_j A_j \leq I \right\}$$

Example: If $C \in M_{2d}$ is given by $C_1 = \mathrm{diag}(1, -1, 0, 0, \dots, 0,)$, $C_2 = \mathrm{diag}(0, 0, 1, -1, 0, \dots, 0)$, $\dots$, $C_d(0, 0, \dots, 0, 1, -1)$, then $\mathcal{D}_C(1) = [-1, 1]^d$.

Spectral inclusion problem: Given A, A', decide whether

$$\mathcal{D}_A(1) \subseteq \mathcal{D}_{A'}(1)$$

Matrix cube problem: Given A as above, decide whether

$$[-1, 1]^d \subseteq \mathcal{D}_A(1)$$

What does operator theory have to say about this?

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Matrix convex sets over the ball

There are many interesting matrix convex sets over the ball $\overline{\mathbb{B}_d} \subseteq \mathbb{R}^d$:

$$\mathcal{W}^{min}(\overline{\mathbb{B}_d}) \quad , \quad \mathcal{W}^{max}(\overline{\mathbb{B}_d})$$

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Haagerup: $\|\sum A_i \otimes \overline{B_i}\| \leq \|\sum A_i \otimes \overline{A_i}\|^{1/2} \|\sum B_i \otimes \overline{B_i}\|^{1/2}$**, so this is a matrix convex set.**

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$\mathfrak{B}(1) = \mathfrak{D}(1) = \overline{\mathbb{B}_d}$.

$$\mathcal{W}^{\text{min}}(\overline{\mathbb{B}_d}) \subset \mathfrak{B} \subset \mathfrak{D} = \mathfrak{D}^\bullet \subset \mathfrak{B}^\bullet \subset \mathcal{W}^{\text{max}}(\overline{\mathbb{B}_d})$$

Here $\mathcal{S}^\bullet = \{X \in \mathbb{S}_d : \sum X_i \otimes Y_i \leq I \text{ for all } Y \in \mathcal{S}\}$ **(matrix polar dual).**

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Let $\mathcal{S} \subseteq \cup_n (M_n)_{sa}^d$ be a matrix convex set. Then

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Corollary

selfadjoint row contraction $\xrightarrow{dilates}$ commuting selfadjoint row contraction (up to scale of $\sqrt{d}$).