NC function theory, NC convex sets, and operator theory
Lecture II: Algebras of bounded analytic nc functions on nc varieties

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Based on a joint work with Guy Salomon and Eli Shamovich

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Noncommutative functions and varieties

The nc ball and nc functions

nc sets

- $\mathcal{M}^d := \bigcup_{n=1}^{\infty} M_n^d$ where $d \in \mathbb{N}$. 

nc functions

- $f$ is graded:

- $f$ respects direct sums:

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  \|X\| := \| \sum_{j=1}^{d} X_j X^*_j \|^{\frac{1}{2}}
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A function $f : \Omega \rightarrow \mathbb{M}^e$ is a nc function if

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Let $E$ be a set of bounded nc functions $\mathcal{B}_d \to \mathbb{M}^1$. 
Let $\mathcal{E}$ be a set of bounded nc functions $\mathcal{B}_d \to \mathbb{M}^1$.

- **The nc (analytic) variety (in the nc unit ball)** generated by $\mathcal{E}$ is

  \[ \mathcal{V} = \mathcal{V}(\mathcal{E}) = \{ X \in \mathcal{B}_d : f(X) = 0 \text{ for all } f \in \mathcal{E} \}. \]
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Example: $H^\infty(\mathcal{B}_d) =$ $H^\infty(C^d)$.
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Example: $H^\infty(\mathcal{B}_d) = H^\infty(B(\mathcal{X})_1^d) = \mathcal{L}_d = H^\infty(\mathbb{C}^d)$.

Recall:

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Classification of the algebras $H^\infty(\mathcal{V})$

The goal:
To classify the algebras $H^\infty(\mathcal{V})$ in terms of the geometry of the variety $\mathcal{V}$. 

Why care:
(i) These algebras (and their kin $A(\mathcal{V})$) have independently drawn interest.
(ii) The problem applies, applies to, and inspires nc function theory.

"Geometry" — certain equivalence of nc varieties $\mathcal{V} \hookrightarrow \mathcal{W} \cong \mathcal{B}^d$.

**Biholomorphism:** there exist nc holomorphic $f: \mathcal{B}^d \to \mathcal{M}^d$ and $g: \mathcal{B}^d \to \mathcal{M}^d$ such that $g \circ f = \text{id}_{\mathcal{V}}$ and $f \circ g = \text{id}_{\mathcal{W}}$.

**Ball-biholomorphism:** there exist nc holomorphic $f: \mathcal{B}^d \to \mathcal{B}^d$ and $g: \mathcal{B}^d \to \mathcal{B}^d$ such that $g \circ f = \text{id}_{\mathcal{V}}$ and $f \circ g = \text{id}_{\mathcal{W}}$.

**Conformal equivalence:** there exists $\gamma \in \text{Aut}(\mathcal{B}^d)$ such that $\gamma(V) = \mathcal{W}$. 

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The isomorphism problem

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- completely isometric isomorphism
- completely bounded isomorphism
- isometric isomorphism
- bounded isomorphism
- weak-∗ continuous isomorphism
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Completely isometric isomorphisms

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Theorem (Salomon-S-Shamovich ’17)

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{B}_d$ be nc varieties. There is a completely isometric isomorphism $\alpha : H^\infty(\mathcal{V}) \to H^\infty(\mathcal{W})$ if and only if $\mathcal{V}$ and $\mathcal{W}$ are ball-biholomorphic.

Classically, a “ball-biholomorphism” is an automorphism. Is this true in the nc setting as well? Need to understand fixed points.

Theorem (Shamovich, later in ’17)

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{B}_d$ be nc varieties which contain a scalar point. If $\mathcal{V}$ and $\mathcal{W}$ are ball-biholomorphic, then $\mathcal{V}$ and $\mathcal{W}$ are conformally equivalent.
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Furthermore, in this case, the isomorphism of the algebras is given by a precomposition with the ball-biholomorphism:

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\alpha(f) = f \circ G , \quad f \in H^\infty(\mathcal{W})
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Proof ingredients

Theorem (Salomon-S-Shamovich, ’17, based on Davidson-Pitts ’98)

\[ \pi : \text{Rep}_{cc}(H^\infty(\mathcal{M})) \rightarrow \overline{\mathcal{B}}_d \]
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\(\pi\) is injective on weak-\(*\) representations and

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Proof: \( z = (z_1, \ldots, z_d) \) is a row contraction.
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**Proof:** \( z = (z_1, \ldots, z_d) \) is a row contraction. If \( \Phi : \mathcal{H}^\infty(\mathcal{M}) \to M_n \) is cc, then

\[ \pi(z) := \Phi(z) = (\Phi(z_1), \ldots, \Phi(z_d)) \in \overline{\mathfrak{B}}_d \]
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Imagine that \( \mathcal{V} \) is cut out by polynomial equations. Then \( z = z|_{\mathcal{V}} \) satisfies all equation determining \( \mathcal{V} \), so morally: \( \pi(z) = \Phi(z) \in \mathcal{V} \).
Proof ingredients (iso. given by composition with map)

\[ \pi : \text{Rep}_{cc}(H^\infty(V)) \to \mathcal{B}_d : \pi(\Phi) = (\Phi(z_1), \ldots, \Phi(z_d)) \in \mathcal{B}_d. \]
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Over \( X \in \mathcal{V} \) lies a unique rep: the weak-* cont. evaluation representation \( \Phi_X : f \mapsto f(X) \).
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If \( \alpha : H^\infty(\mathcal{V}) \to H^\infty(\mathcal{W}) \) is a c.i.i., define \( G : \mathcal{W} \to \overline{\mathcal{B}}_d \) by

\[ G(W) = \pi \alpha^*(\Phi_W) \]
Completely isometric isomorphisms

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By the nc extension theorem, \( G \) extends to \( G : \mathcal{B}_d \to \overline{\mathcal{B}}_d \), and in fact, by nc maximum principle \( G(\mathcal{B}_d) \subseteq \mathcal{B}_d \).
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It follows that \( \alpha^*(\Phi_W) = \Phi_{G(W)} \),
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Completely isometric isomorphisms

Proof ingredients (iso. given by composition with map)

\[ \pi : \text{Rep}_{cc}(H^\infty(\mathcal{M})) \to \mathfrak{B}_d : \pi(\Phi) = (\Phi(z_1), \ldots, \Phi(z_d)) \in \mathfrak{B}_d. \]

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By some nc function theory, \( G \) extends to \( G : \mathcal{B}_d \to \overline{\mathcal{B}}_d \), and in fact \( G(\mathcal{B}_d) \subseteq \mathcal{B}_d \).

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The isomorphism problem – **homogeneous case**

- Conformal equivalence
- Ball-biholomorphism
- Completely isometric isomorphism
- Completely bounded isomorphism
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Theorem (Salomon-S-Shamovich ’17)

Let $\mathcal{V} \subseteq \mathcal{B}_d$ and $\mathcal{W} \subseteq \mathcal{B}_e$ be homogeneous nc varieties. Then TFAE:

- $H_1(\mathcal{V})$ and $H_1(\mathcal{W})$ are completely isometrically isomorphic,
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**Proof ingredients:**

- Ball-biholo, conf. equiv. — basic nc function theory and tricks, analysis of fixed points of nc holomorphic maps (refined by Shamovich later).
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In search of a coarser classification of $H^\infty(\mathcal{V})$
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- Can we classify up to c.b./bounded/weak-\* cont./algebraic isomorphism?
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- Can we classify up to c.b./bounded/weak-\*$\text{-cont.}/algebraic isomorphism?
- We need new types of variety equivalences in our story...
The similarity envelope of a nc set $\Omega$ is

$$\tilde{\Omega} := \bigcup_{n=1}^{\infty} \left\{ S^{-1}XS : X \in \Omega(n), \ S \in \text{GL}_n(\mathbb{C}) \right\}.$$ 

$\tilde{\Omega}$ is a nc set.
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*Every nc function on an nc set $\Omega$ extends uniquely to $\tilde{\Omega}$.***
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$$\tilde{f}(S^{-1}XS) := S^{-1}f(X)S$$

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Thus, $H^\infty(\mathcal{V})$ is an algebra of (unbounded) nc functions on $\tilde{\Omega}$. 

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Example: the similarity envelope of $\mathcal{N} = \mathcal{B}_d$

Recall:
$\mathcal{B}_d$ is the set of all strict contractions. $\tilde{\mathcal{B}}_d$ is the similarity envelope of $\mathcal{B}_d$. What does $\tilde{\mathcal{B}}_d$ look like?
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$\tilde{\mathcal{B}}_d$ is the set of all pure $d$-tuples.
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The **joint spectral radius** is defined by $\rho(X) = \lim \|\Psi_X^k(I_n)\|^{\frac{1}{2k}}$. 
Example (cont.): $\text{Aut}(\mathcal{B}_d)$

**Theorem** (Follows from work of many: Davison-Pitts, Popescu ... explicitly: McCarthy-Timoney, us)

$$\text{Aut}(\mathcal{B}_d) = \text{Aut}(\mathcal{B}_d)$$
Similarity envelopes

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Clear: \( \text{Aut}(\mathcal{B}_d) \hookrightarrow \text{Aut}(\mathcal{\tilde{B}}_d) \). But is it all?

**Example**

If \( g \in H^\infty(\mathcal{B}_d) \) is invertible, the nc map

\[
G(X) = g(X)Xg(X)^{-1}
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is a nc automorphism of \( \mathcal{\tilde{B}}_d \).
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$$f \mapsto gfg^{-1} = f \circ G$$

* One goal: understand automorphisms of $H^\infty(\mathcal{B}_d)$. Time permitting, we'll discuss.
The role of similarity envelope — finite dim. reps.

Theorem (Salomon-S-Shamovich, '18)

\[ \pi : \text{Rep}_b(H^\infty(\mathcal{W})) \to \overline{\mathcal{B}}_d \]
\[ \text{Rep}_{w^*}(H^\infty(\mathcal{W})) \cong \overline{\mathcal{Y}} \]
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Proof: If \( \Phi : H^\infty(\mathcal{V}) \rightarrow M_n \) is bounded
The role of similarity envelope — finite dim. reps.

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The role of similarity envelope — finite dim. reps.

**Theorem (Salomon-S-Shamovich, ’18)**

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**Proof:** If \( \Phi : H^{\infty}(\mathcal{V}) \to M_n \) is bounded \( \xrightarrow{\text{Smith}} \) \( \Phi \) is cb \( \xrightarrow{\text{Paulsen}} \) \( \Phi \) is similar to cc representation \( \Phi(\cdot) = S^{-1}\Psi(\cdot)S \) with \( \Psi \in \text{Rep}_{cc}(H^{\infty}(\mathcal{V})) \).
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$$\pi : \Phi \mapsto \pi(\Phi) = (\Phi(z_1), \ldots, \Phi(z_d)) = S^{-1} \Psi(z) S$$

maps $$\text{Rep}_b(H^\infty(\mathcal{V}))$$ onto similarity envelope of $$\pi(\text{Rep}_{cc}(H^\infty(\mathcal{V})))$$. 
The role of similarity envelope — finite dim. reps.

**Theorem (Salomon-S-Shamovich, ’18)**

\[ \pi : \text{Rep}_b(H^\infty(\mathcal{H})) \to \tilde{\mathcal{B}}_d \]

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maps \( \text{Rep}_b(H^\infty(\mathcal{H})) \) onto similarity envelope of \( \pi(\text{Rep}_{cc}(H^\infty(\mathcal{H}))) \). \( \Phi \) is weak-* continuous \( \Leftrightarrow \Psi \) is, so second assertion follows. \( \square \)
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**Evaluation representations**

For \( X \in \tilde{\mathcal{H}} \) define \( \Phi_X \in \text{Rep}_{w^*}(H^\infty(\mathcal{H})) \) by

\[ \Phi_X(f) = \tilde{f}(X). \]
General strategy

Suppose $\alpha : H^\infty(\mathcal{V}) \to H^\infty(\mathcal{W})$ is a bounded isomorphism.
Suppose \( \alpha : H^\infty(\mathcal{H}) \to H^\infty(\mathcal{M}) \) is a bounded isomorphism.

\[ \implies \text{we have the adjoint map } \alpha^* : H^\infty(\mathcal{M}) \to H^\infty(\mathcal{H}). \]
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Define \( G : \mathcal{W} \to \mathcal{B}_d \) by

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G(W) = \pi \alpha^*(\Phi_W)
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$\implies$ define $G : \mathfrak{W} \to \mathfrak{B}_d$ by 

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Over $X \in \mathfrak{H}$ lies a unique rep: the weak-* cont. evaluation representation 

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\[ \implies \text{define } G : \mathfrak{M} \to \mathfrak{B}_d \text{ by } \]

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Over \( X \in \tilde{\mathfrak{H}} \) lies a unique rep: the weak-* cont. evaluation representation

\[ \Phi_X : f \mapsto \tilde{f}(X) \]

Our goal:

To show that \( G(W) \in \tilde{\mathfrak{H}} \) for every \( W \in \mathfrak{M} \)

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$$\alpha(f)(W) = \alpha^*(\Phi_W)(f) = \Phi_{G(W)}(f) = \tilde{f} \circ G(W)$$

Note: working backwards, get boundedness condition on $G$:

$$\|\Phi_{G(W)}(f)\| = \|\tilde{f} \circ G(W)\| = \|\alpha(f)(W)\| \leq \|\alpha(f)\| \leq \|\alpha\|f\|$$
Easy result: weak-\(\ast\) continuous isomorphisms

\[ \tilde{\mathcal{V}} := \bigsqcup_{n=1}^{\infty} \{ S^{-1}XS : X \in \mathcal{V}(n), \ S \in \text{GL}_n(\mathbb{C}) \} . \]
Easy result: weak-\ast continuous isomorphisms

\[ \tilde{\mathcal{V}} := \bigsqcup_{n=1}^{\infty} \{ S^{-1}XS : X \in \mathfrak{V}(n), \ S \in \text{GL}_n(\mathbb{C}) \} . \]

\[ \tilde{\mathcal{V}} \leftrightarrow \text{Rep}_{w^*}(H^\infty(\mathfrak{V})) \ \text{by} \ \ X \leftrightarrow \Phi_X \]
Easy result: weak-\(\ast\) continuous isomorphisms

\[
\mathcal{V} := \bigsqcup_{n=1}^{\infty} \{ S^{-1} X S : X \in \mathcal{V}(n), \ S \in \text{GL}_n(\mathbb{C}) \}.
\]

\[
\mathcal{V} \longmapsto \text{Rep}_{w^\ast}(H^\infty(\mathcal{V})) \quad \text{by} \quad X \longmapsto \Phi_X : f \mapsto \hat{f}(X).
\]
Bounded isomorphisms

Easy result: weak-* continuous isomorphisms

\[ \tilde{\mathcal{V}} := \bigsqcup_{n=1}^{\infty} \{ S^{-1} X S : X \in \mathfrak{V}(n), \ S \in \text{GL}_n(\mathbb{C}) \} . \]

\[ \tilde{\mathcal{V}} \longleftrightarrow \text{Rep}_{w^*}(H^\infty(\mathcal{V})) \ by \ X \longleftrightarrow \Phi_X : f \mapsto \tilde{f}(X). \]

Theorem (Salomon-S-Shamovich '18)

Let \( \mathcal{V}, \mathcal{W} \subseteq \mathcal{B}_d \) be nc varieties. Then \( H^\infty(\mathcal{V}) \) and \( H^\infty(\mathcal{W}) \) are weak-* isomorphic if and only if \( \tilde{\mathcal{V}} \) and \( \tilde{\mathcal{W}} \) are biholomorphic via a nc map \( G : \tilde{\mathcal{W}} \rightarrow \tilde{\mathcal{V}} \) satisfying

\[ \sup_{W \in \mathcal{W}} \| \Phi_{G(W)} \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi_{G^{-1}(V)} \| < \infty \]

where \( \Phi_X = \text{evaluation at } X \).
Bounded isomorphisms

Easy result: weak-\,* continuous isomorphisms

$$\tilde{\mathcal{V}} := \bigsqcup_{n=1}^{\infty} \{ S^{-1} X S : X \in \mathfrak{V}(n), \ S \in \text{GL}_n(\mathbb{C}) \}.$$  

$$\tilde{\mathcal{V}} \longleftrightarrow \text{Rep}_{w^\ast}(H^\infty(\mathcal{V})) \text{ by } X \longleftrightarrow \Phi_X : f \mapsto \tilde{f}(X).$$

Theorem (Salomon-S-Shamovich ’18)

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{B}_d$ be nc varieties. Then $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ are weak-\,* isomorphic if and only if $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$ are biholomorphic via a nc map $G : \tilde{\mathcal{W}} \to \tilde{\mathcal{V}}$ satisfying

$$\sup_{W \in \mathcal{W}} \| \Phi_{G(W)} \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi_{G^{-1}(V)} \| < \infty$$

where $\Phi_X = \text{evaluation at } X$. In this case, the isomorphism of $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ is given by

$$f \mapsto \tilde{f} \circ G$$
Bounded isomorphisms

Proving iso. ⇒ biholo. in the easy result

Theorem (Salomon-S-Shamovich ’18)

Let $V, W \subseteq \mathcal{B}_d$ be nc varieties. Then $H^\infty(V)$ and $H^\infty(W)$ are weak-$\ast$ isomorphic if and only if $\tilde{V}$ and $\tilde{W}$ are biholomorphic via a nc map $G : \tilde{W} \to \tilde{V}$ satisfying

$$\sup_{W \in \tilde{W}} \| \Phi_G(W) \| < \infty \quad \text{and} \quad \sup_{V \in \tilde{V}} \| \Phi_{G^{-1}}(V) \| < \infty$$

The isomorphism of is given by $f \mapsto \tilde{f} \circ G$.

Proof (iso. ⇒ biholo.):

Recall that we defined $G : f : W \to f : \mathcal{B}_d$ by $G(W) = \pi_{\Phi}^\ast(W)$. We need to prove that $G$ maps $W$ into $V$, but this is clear if $\pi$ is weak-$\ast$ continuous (given that $\pi : W \to \overline{\pi W} = \text{Rep}_w(\overline{H^1(V)})$).
Theorem (Salomon-S-Shamovich ’18)

Let $V, \mathcal{M} \subseteq \mathcal{B}_d$ be nc varieties. Then $H_\infty(\mathcal{V})$ and $H_\infty(\mathcal{M})$ are weak-$\ast$ isomorphic if and only if $\mathcal{V}$ and $\mathcal{M}$ are biholomorphic via a nc map $G: \mathcal{M} \rightarrow \mathcal{V}$ satisfying

$$\sup_{W \in \mathcal{M}} \|\Phi_G(W)\| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \|\Phi_{G^{-1}}(V)\| < \infty$$

The isomorphism of is given by $f \mapsto \tilde{f} \circ G$.

Proof (iso. $\Rightarrow$ biholo.): Recall that we defined $G: \mathcal{M} \rightarrow \mathcal{B}_d$ by

$$G(W) = \pi \alpha^*(\Phi_W)$$
**Theorem (Salomon-S-Shamovich ’18)**

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{B}_d$ be nc varieties. Then $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ are weak-* isomorphic if and only if $\mathcal{V}$ and $\mathcal{W}$ are biholomorphic via a nc map $G : \mathcal{W} \to \mathcal{V}$ satisfying

$$
\sup_{W \in \mathcal{W}} \| \Phi_G(W) \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi_{G^{-1}}(V) \| < \infty
$$

The isomorphism of is given by $f \mapsto f \circ G$.

**Proof (iso. $\Rightarrow$ biholo.):** Recall that we defined $G : \mathcal{W} \to \mathcal{B}_d$ by

$$
G(W) = \pi \alpha^*(\Phi_W)
$$

We need to prove that $G$ maps $\mathcal{W}$ into $\mathcal{V}$, but this is clear if $\alpha$ is weak-* continuous (given that $\mathcal{V} \cong \text{Rep}_{w^*}(H^\infty(\mathcal{V}))$).
Theorem (Salomon-S-Shamovich ’18)

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{B}_d$ be nc varieties. Then $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ are weak-\$
\begin{align*}
\text{isomorphic} \text{ if and only if } \tilde{\mathcal{V}} \text{ and } \tilde{\mathcal{W}} \text{ are biholomorphic via a nc map } \\
G : \tilde{\mathcal{W}} \to \tilde{\mathcal{V}} \text{ satisfying }
\end{align*}

\[ \sup_{W \in \mathcal{W}} \| \Phi_G(W) \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi_G^{-1}(V) \| < \infty \]

The isomorphism of is given by $f \mapsto \tilde{f} \circ G$.

Proof (bounded biholo. $\Rightarrow$ iso.):
Theorem (Salomon-S-Shamovich ’18)

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{B}_d$ be nc varieties. Then $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ are weak-\*$ isomorphic if and only if $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$ are biholomorphic via a nc map $G : \tilde{\mathcal{W}} \to \tilde{\mathcal{V}}$ satisfying

$$\sup_{W \in \mathcal{W}} \| \Phi_G(W) \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi_{G^{-1}}(V) \| < \infty$$

The isomorphism is given by $f \mapsto \tilde{f} \circ G$.

Proof (bounded biholo. $\Rightarrow$ iso.): For $f \in H^\infty(\mathcal{V})$,

$$\sup_{W \in \mathcal{W}} \| \tilde{f}(G(W)) \| \leq \sup_{W \in \mathcal{W}} \| \Phi_G(W) \| \| f \|,$$

so $\tilde{f} \circ G \in H^\infty(\mathcal{W})$. 
Bounded isomorphisms

Proving the other easy direction of the easy result

Theorem (Salomon-S-Shamovich ’18)

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{V}_d$ be nc varieties. Then $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ are weak-$\star$ isomorphic if and only if $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$ are biholomorphic via a nc map $G : \tilde{\mathcal{W}} \rightarrow \tilde{\mathcal{V}}$ satisfying

$$
\sup_{W \in \mathcal{W}} \| \Phi_G(W) \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi_{G^{-1}}(V) \| < \infty
$$

The isomorphism of is given by $f \mapsto \tilde{f} \circ G$.

Proof (bounded biholo. $\Rightarrow$ iso.): For $f \in H^\infty(\mathcal{V})$,

$$
\sup_{W \in \mathcal{W}} \| \tilde{f}(G(W)) \| \leq \sup_{W \in \mathcal{W}} \| \Phi_G(W) \| \| f \|,
$$

so $\tilde{f} \circ G \in H^\infty(\mathcal{W})$.

$\alpha : f \mapsto \tilde{f} \circ G$ well defined and bounded $\Rightarrow$ homomorphism.
Removing the weak-* assumption

Goal: show $G = \pi \circ \alpha^*$ maps $\mathcal{W}$ to $\tilde{\mathcal{V}}$. 
Goal: show $G = \pi \circ \alpha^*$ maps $\mathcal{W}$ to $\mathcal{Y}$, i.e. $G(W) \in \mathcal{Y}$ for $W \in \mathcal{W}$.
Goal: show $G = \pi \circ \alpha^*$ maps $\mathcal{W}$ to $\tilde{\mathcal{V}}$, i.e. $G(W) \in \tilde{\mathcal{V}}$ for $W \in \mathcal{W}$.

Since $G(W) \in \tilde{\mathcal{V}}$, we have $G(W) \in \tilde{\mathcal{V}} \iff \rho(G(W)) < 1$. 
Removing the weak-\(\ast\) assumption

**Goal:** show \(G = \pi \circ \alpha^*\) maps \(\mathcal{W}\) to \(\tilde{\mathcal{Y}}\), i.e. \(G(W) \in \tilde{\mathcal{Y}}\) for \(W \in \mathcal{W}\).

Since \(G(W) \in \tilde{\mathcal{Y}}\), we have \(G(W) \in \tilde{\mathcal{Y}} \iff \rho(G(W)) < 1\).

We believe this is true in general, but could prove it (for now) only for homogeneous varieties.
Goal: show $G = \pi \circ \alpha^*$ maps $\mathcal{W}$ to $\tilde{\mathcal{V}}$, i.e. $G(W) \in \tilde{\mathcal{V}}$ for $W \in \mathcal{W}$. Since $G(W) \in \tilde{\mathcal{V}}$, we have $G(W) \in \tilde{\mathcal{V}} \iff \rho(G(W)) < 1$.

We believe this is true in general, but could prove it (for now) only for homogeneous varieties.

Theorem (Salomon-S-Shamovich '18)

Let $\mathcal{V}$, $\mathcal{W} \subseteq \mathcal{B}_d$ be homogeneous nc varieties. Then $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ are boundedly isomorphic if and only if $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$ are biholomorphic via a nc map $G : \tilde{\mathcal{W}} \rightarrow \tilde{\mathcal{V}}$ satisfying

\[
\sup_{W \in \mathcal{W}} \| \Phi G(W) \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi G^{-1}(V) \| < \infty
\]

The isomorphism of is given by $f \mapsto \tilde{f} \circ G$. 

Theorem (Salomon-S-Shamovich '18)
Removing the weak-* assumption

**Theorem (Salomon-S-Shamovich ’18)**

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{V}_d$ be homogeneous nc varieties. Then $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ are **boundedly** isomorphic if and only if $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$ are biholomorphic via a nc map $G : \tilde{\mathcal{W}} \to \tilde{\mathcal{V}}$ satisfying

$$
\sup_{W \in \mathcal{W}} \| \Phi_G(W) \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi_{G^{-1}}(V) \| < \infty
$$

The isomorphism of is given by $f \mapsto \tilde{f} \circ G$.

**Proof idea:** $G = \pi \circ \alpha^*$. Need to show $\rho(G(W)) < 1$ for $W \in \mathcal{W}$. 
Bounded isomorphisms

Removing the weak-* assumption

**Theorem (Salomon-S-Shamovich ’18)**

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{B}_d$ be homogeneous nc varieties. Then $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ are boundedly isomorphic if and only if $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{W}}$ are biholomorphic via a nc map $G : \widetilde{\mathcal{W}} \to \widetilde{\mathcal{V}}$ satisfying

$$\sup_{W \in \mathcal{W}} \| \Phi_G(W) \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi_{G^{-1}}(V) \| < \infty$$

**The isomorphism of is given by** $f \mapsto \tilde{f} \circ G$.

**Proof idea:** $G = \pi \circ \alpha^*$. Need to show $\rho(G(W)) < 1$ for $W \in \mathcal{W}$.

Define $u : \mathbb{D} \to [0, \infty)$ by $u(z) = \rho(G(zW/\|W\|))$. 

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Removing the weak-* assumption

**Theorem (Salomon-S-Shamovich ’18)**

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{B}_d$ be homogeneous nc varieties. Then $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ are **boundedly isomorphic** if and only if $\mathcal{V}$ and $\mathcal{W}$ are biholomorphic via a nc map $G : \mathcal{W} \to \mathcal{V}$ satisfying

$$ \sup_{W \in \mathcal{W}} \| \Phi_{G(W)} \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi_{G^{-1}(V)} \| < \infty $$

The isomorphism of is given by $f \mapsto \widetilde{f} \circ G$.

**Proof idea:** $G = \pi \circ \alpha^*$. Need to show $\rho(G(W)) < 1$ for $W \in \mathcal{W}$. Define $u : \mathbb{D} \to [0, \infty)$ by $u(z) = \rho(G(zW/\|W\|))$. We prove (following Vesentini) that this function $u$ is subharmonic, thus satisfies a maximum principle.
Theorem (Salomon-S-Shamovich ’18)

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{B}_d$ be homogeneous nc varieties. Then $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ are boundedly isomorphic if and only if $\overline{\mathcal{V}}$ and $\overline{\mathcal{W}}$ are biholomorphic via a nc map $G : \mathcal{W} \to \mathcal{V}$ satisfying

$$\sup_{W \in \mathcal{W}} \| \Phi_{G(W)} \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi_{G^{-1}(V)} \| < \infty$$

The isomorphism of is given by $f \mapsto \tilde{f} \circ G$.

Proof idea: $G = \pi \circ \alpha^*$. Need to show $\rho(G(W)) < 1$ for $W \in \mathcal{W}$. Define $u : \mathbb{D} \to [0, \infty)$ by $u(z) = \rho(G(zW/\|W\|))$. We prove (following Vesentini) that this function $u$ is subharmonic, thus satisfies a maximum principle. If $u(\|W\|) = \rho(G(W)) = 1$, then $u \equiv 1$. 
Removing the weak-* assumption

**Theorem (Salomon-S-Shamovich ’18)**

Let $V, W \subseteq \mathcal{B}_d$ be homogeneous nc varieties. Then $H^\infty(V)$ and $H^\infty(W)$ are **boundedly** isomorphic if and only if $\tilde{V}$ and $\tilde{W}$ are biholomorphic via a nc map $G : \tilde{W} \to \tilde{V}$ satisfying

$$\sup_{W \in \mathcal{W}} \| \Phi_{G(W)} \| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \| \Phi_{G^{-1}(V)} \| < \infty$$

The isomorphism of is given by $f \mapsto \tilde{f} \circ G$.

**Proof idea:** $G = \pi \circ \alpha^*$. Need to show $\rho(G(W)) < 1$ for $W \in \mathcal{W}$. Define $u : \mathbb{D} \to [0, \infty)$ by $u(z) = \rho(G(zW/\|W\|))$. We prove (following Vesentini) that this function $u$ is subharmonic, thus satisfies a maximum principle. If $u(\|W\|) = \rho(G(W)) = 1$, then $u \equiv 1$. Thus $u(0) = \rho(G(0)) = 1$,
Removing the weak-* assumption

**Theorem (Salomon-S-Shamovich ’18)**

Let $\mathcal{V}, \mathcal{W} \subseteq \mathcal{B}_d$ be homogeneous nc varieties. Then $H^\infty(\mathcal{V})$ and $H^\infty(\mathcal{W})$ are **boundedly isomorphic** if and only if $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{W}}$ are biholomorphic via a nc map $G : \tilde{\mathcal{W}} \to \tilde{\mathcal{V}}$ satisfying

$$\sup_{W \in \mathcal{W}} \|\Phi_{G(W)}\| < \infty \quad \text{and} \quad \sup_{V \in \mathcal{V}} \|\Phi_{G^{-1}(V)}\| < \infty$$

The isomorphism of is given by $f \mapsto \tilde{f} \circ G$.

**Proof idea:** $G = \pi \circ \alpha^*$. Need to show $\rho(G(W)) < 1$ for $W \in \mathcal{W}$. Define $u : \mathbb{D} \to [0, \infty)$ by $u(z) = \rho(G(zW/\|W\|))$. We prove (following Vesentini) that this function $u$ is subharmonic, thus satisfies a maximum principle. If $u(\|W\|) = \rho(G(W)) = 1$, then $u \equiv 1$. Thus $u(0) = \rho(G(0)) = 1$, and all of $\mathcal{W}$ is mapped to the boundary, leading to a contradiction.
We define a pseudometric on $\mathcal{B}_d$ by

$$\delta(X, Y) := \sup_{f \in (H^\infty(\mathcal{B}_d))_1} \|f(X^n) - f(Y^m)\|$$

for $X \in \mathcal{B}_d(m)$ and $Y \in \mathcal{B}_d(n)$. 
We define a **pseudometric** on $\mathcal{B}_d$ by

$$\delta(X, Y) := \sup_{f \in (H^\infty(\mathcal{B}_d))_1} \| f(X^{(n)}) - f(Y^{(m)}) \|$$

for $X \in \mathcal{B}_d(m)$ and $Y \in \mathcal{B}_d(n)$.

A new type of variety equivalence:
We define a pseudometric on $\mathcal{B}_d$ by

$$\delta(X, Y) := \sup_{f \in (H^\infty(\mathcal{B}_d))_1} \| f(X^{(n)}) - f(Y^{(m)}) \|$$

for $X \in \mathcal{B}_d(m)$ and $Y \in \mathcal{B}_d(n)$.

A new type of variety equivalence:

We say that $\mathcal{V}$ & $\mathcal{W}$ are bi-Lipschitz equivalent if their similarity envelopes $\tilde{V}$ and $\tilde{W}$ are biholomorphic via a bi-Lipschitz map.
We define a **pseudometric** on $\mathcal{B}_d$ by

$$\delta_{cb}(X, Y) := \sup_k \sup_{f \in (M_k(H^\infty(\mathcal{B}_d)))_1} \| f(X^{(n)}) - f(Y^{(m)}) \|$$

for $X \in \mathcal{B}_d(m)$ and $Y \in \mathcal{B}_d(n)$.

A new type of variety equivalence:

We say that $\mathcal{V}$ & $\mathcal{W}$ are **completely bi-Lipschitz equivalent** if their similarity envelopes $\mathcal{V}$ and $\mathcal{W}$ are biholomorphic via a bi-Lipschitz map.
Tidying up

For a nc holomorphic map $G : \mathfrak{W} \to \mathfrak{Y}$ TFAE:

(i) $\sup_{W \in \mathfrak{W}} \| \Phi_G(W) \| < \infty$

(ii) $G$ is a Lipschitz map w.r.t. $\delta$
For a nc holomorphic map $G : \tilde{\mathcal{W}} \rightarrow \tilde{\mathcal{V}}$ TFAE:

(i) $\sup_{W \in \tilde{\mathcal{W}}} \| \Phi_G(W) \| < \infty$

(ii) $G$ is a Lipschitz map w.r.t. $\delta$

Also, TFAE:

(i) $\sup_{W \in \tilde{\mathcal{W}}} \| \Phi_G(W) \|_{cb} < \infty$

(ii) $G$ is a Lipschitz map w.r.t. $\delta_{cb}$
Homogeneous varieties – cb and bounded isomorphisms
Homogeneous varieties – cb and bounded isomorphisms

- Conformal equivalence
- Ball-biholomorphism
- Completely bi-Lipschitz equivalence
- Bi-Lipschitz equivalence
- Algebraic isomorphism
- Bounded isomorphism
- Completely bounded isomorphism
- Completely isometric isomorphism
- Isometric isomorphism
- Weak-* continuous isomorphism
Homogeneous varieties – cb and bounded isomorphisms

- Conformal equivalence
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Homogeneous varieties – cb and bounded isomorphisms
Homogeneous varieties – cb and bounded isomorphisms
Using some complex analysis techniques, we can also show (only for **homogeneous** varieties):

\[ bL\text{-equiv} \]
More on homogeneous varieties

Using some complex analysis techniques, we can also show (only for homogeneous varieties):

\[ \text{bL-equiv} \rightsquigarrow \text{0-preserving bL-equiv} \]
Using some complex analysis techniques, we can also show (only for homogeneous varieties):

\[
bL\text{-equiv} \leadsto \text{0-preserving } bL\text{-equiv} \leadsto \text{linear } bL\text{-equiv}
\]
Homogeneous varieties

- Conformal equivalence
- Ball-biholomorphism
- Completely bi-Lipschitz equivalence
- Bi-Lipschitz equivalence
- Completely bi-Lipschitz linear equivalence
- Completely bounded isomorphism
- Bounded isomorphism
- Algebraic isomorphism
- Weak-* continuous isomorphism
- Completely isometric isomorphism
Homogeneous varieties

- conformal equivalence
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- completely isometric isomorphism
- isometric isomorphism
Theorem (S–Shalit–Shamovich, 2018)

Let $\mathcal{V} \subseteq \mathcal{B}_d$ and $\mathcal{W} \subseteq \mathcal{B}_d$ be homogeneous nc varieties. TFAE:

\[
\begin{align*}
H^\infty(\mathcal{V}) & \cong_{cb} H^\infty(\mathcal{W}), \\
H^\infty(\mathcal{V}) & \cong_{b} H^\infty(\mathcal{W}), \\
H^\infty(\mathcal{V}) & \cong_{w^*} H^\infty(\mathcal{W}), \\
\mathcal{V} & \sim_{cbL} \mathcal{W}, \\
\mathcal{V} & \sim_{bL} \mathcal{W},
\end{align*}
\]

∃ completely bi-Lipschitz $A \in \text{GL}_d(\mathbb{C})$ s.t. $A\mathcal{W} = \mathcal{V}$

∃ bi-Lipschitz $A \in \text{GL}_d(\mathbb{C})$ s.t. $A\mathcal{W} = \mathcal{V}$
Bounded isomorphisms

Classification in the case of homogeneous varieties

Theorem (S–Shalit–Shamovich, 2018)

Let $\mathcal{V} \subseteq \mathcal{B}_d$ and $\mathcal{W} \subseteq \mathcal{B}_d$ be homogeneous nc varieties. TFAE:

- $H^\infty(\mathcal{V}) \cong_{cb} H^\infty(\mathcal{W})$,  
- $H^\infty(\mathcal{V}) \cong_b H^\infty(\mathcal{W})$,  
- $H^\infty(\mathcal{V}) \cong_{w^*} H^\infty(\mathcal{W})$,  
- $\tilde{\mathcal{V}} \sim_{cbL} \tilde{\mathcal{W}}$,  
- $\tilde{\mathcal{V}} \sim_{bL} \tilde{\mathcal{W}}$,  

$\exists$ completely bi-Lipschitz $A \in \text{GL}_d(\mathbb{C})$ s.t. $A\tilde{\mathcal{M}} = \tilde{\mathcal{V}}$

$\exists$ bi-Lipschitz $A \in \text{GL}_d(\mathbb{C})$ s.t. $A\tilde{\mathcal{M}} = \tilde{\mathcal{V}}$

Problem

Do we really need to require $A$ to be bi-Lipschitz? After all, it is just an invertible linear map on $\mathbb{C}^d$ ...
That’s all for now . . .
The next talk is independent of the first two!

References - G. Salomon, OS and E. Shamovich

1. “Algebras of bounded noncommutative analytic functions on subvarieties of the noncommutative unit ball", to appear in TAMS.

\[ \mathcal{V}, \mathcal{W} \subseteq \mathcal{C}M^d = \{ X \in M^d : X_iX_j = X_jX_i \text{ for all } i, j \}. \]
Commutative nc setting

\[ \mathcal{V}, \mathcal{W} \subseteq \mathbb{C}\mathbb{M}^d = \{ X \in \mathbb{M}^d : X_i X_j = X_j X_i \text{ for all } i, j \} . \]

If \( V = V(1) \) and \( \mathcal{V} \) is minimal, then \( H^\infty(\mathcal{V}) = \mathcal{M}_V = \text{Mult}(H^2_d) \big|_V \) from first lecture.

Hope this framework may shed light on the isomorphism problem in the fully commutative case.

Watch out. We wish to prove that a linear \( A_2 \in \text{GL}(d) \), with \( A f \mid \mathcal{W} \) is automatically bi-Lipschitz. If we could do that even just for the special case of minimal commutative homogeneous varieties, we would obtain an elegant result of Hartz from the fully commutative case.
Commutative nc setting

\[ \mathcal{V}, \mathcal{W} \subseteq \mathbb{C} \mathbb{M}^d = \{ X \in \mathbb{M}^d : X_iX_j = X_jX_i \text{ for all } i, j \}. \]

If \( V = \mathcal{V}(1) \) and \( \mathcal{V} \) is minimal, then \( H^\infty(\mathcal{V}) = \mathcal{M}_V = \text{Mult}(H^2_d)|_V \) from first lecture.

Hope

This framework may shed light on the isomorphism problem in the fully commutative case.
Commutative nc setting

\[ \mathcal{V}, \mathcal{W} \subseteq \mathcal{C}M^d = \{ X \in M^d : X_iX_j = X_jX_i \text{ for all } i, j \}. \]

If \( V = \mathcal{V}(1) \) and \( \mathcal{V} \) is minimal, then \( H^\infty(\mathcal{V}) = \mathcal{M}_V = \text{Mult}(H^2_d)|_V \) from first lecture.

Hope

This framework may shed light on the isomorphism problem in the fully commutative case.

Watch out

We wish to prove that a linear \( A \in GL_d, \) with \( A\mathcal{W} = \tilde{\mathcal{V}} \) is automatically bi-Lipschitz.
Commutative nc setting

$$\mathcal{V}, \mathcal{W} \subseteq \mathbb{C}M^d = \{ X \in M^d : X_iX_j = X_jX_i \text{ for all } i, j \}.$$

If $V = \mathcal{V}(1)$ and $\mathcal{V}$ is minimal, then $H^\infty(\mathcal{V}) = \mathcal{M}_V = \text{Mult}(H^2_d)|_V$ from first lecture.

Hope

This framework may shed light on the isomorphism problem in the fully commutative case.

Watch out

We wish to prove that a linear $A \in \text{GL}_d$, with $A\tilde{\mathcal{W}} = \tilde{\mathcal{V}}$ is automatically bi-Lipschitz.

If we could do that even just for the special case of minimal commutative homogeneous varieties, we would obtain a new proof of a magnificent result of Hartz from the fully commutative case.
What we know about general varieties

conformal equivalence → ball-biholomorphism
completely bi-Lipschitz equivalence ← bi-Lipschitz equivalence

weak-\* continuous bounded isomorphism → weak-\* continuous completely bounded isomorphism
completely isometric isomorphism ← completely isometric isomorphism

algebraic isomorphism ← isometric isomorphism
Automorphisms of $H^\infty(\mathcal{B}_d)$

**Theorem (Davidson & Pitts, 1998)**

*Every algebraic automorphism of $H^\infty(\mathcal{B}_d)$ is weak-\(^*\) continuous.*
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**Corollary (Salomon-S-Shamovich ’18)**

Every automorphism $\alpha$ of $H^\infty(\mathfrak{B}_d)$ has the form

$$\alpha(f) = \tilde{f} \circ G, \ f \in H^\infty(\mathfrak{B}_d)$$

for some $G \in \text{Aut}_b(\mathfrak{B}_d)$. 
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\[
\begin{align*}
\text{Aut}_b(\tilde{\mathcal{B}}_d) = & \\
\left\{ G \in \text{Aut}(\tilde{\mathcal{B}}_d) : \sup_{X \in \mathcal{B}_d} \| \Phi_G(X) \| < \infty , \quad \sup_{X \in \mathcal{B}_d} \| \Phi_{G^{-1}}(X) \| < \infty \right\}. 
\end{align*}
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- We don’t know what are the (bounded) automorphisms of $\tilde{\mathcal{B}}_d$.
- $\text{Aut}_b(\tilde{\mathcal{B}}_d)$ is larger than $\text{Aut}(\mathcal{B}_d) = \text{Möbius transformations}$, e.g., for each $g \in H^\infty(\mathcal{B}_d)$ invertible, $X \mapsto g(X)^{-1}Xg(X)$ is an automorphism of $\mathcal{B}_d$ that fixes $\mathcal{B}_d(1)$. 
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- Are these all? We don’t know whether Möbius transformations and “inner” automorphisms generate $\text{Aut}_b(\widetilde{\mathcal{B}}_d)$. 
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- Are these all? We don’t know whether Möbius transformations and “inner” automorphisms generate $\text{Aut}_b(\tilde{\mathcal{B}}_d)$.
- We do know that $\text{Aut}_b(\tilde{\mathcal{B}}_d) \subsetneq \text{Aut}(\tilde{\mathcal{B}}_d)$, e.g. take $g$ unbounded.
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Note: $\alpha$ inner $\iff$ $G$ inner, in the sense that

$$G(X) = g(X)Xg(X)^{-1}$$

for some $g \in H^\infty(\mathcal{B}_d)$. 

Quasi-inner atomorphisms are almost inner

**Corollary (Salomon-S-Shamovich '18)**

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**Theorem (Salomon-S-Shamovich '18)**

Let $\alpha: H^\infty(\mathcal{B}_d) \to H^\infty(\mathcal{B}_d)$ be a quasi-inner automorphism, given by

$$\alpha(f) = \tilde{f} \circ G$$

Then for every irreducible $X \in \mathcal{B}_d$, $G(X)$ is similar to $X$. 

That’s really all for now.