NC function theory, NC convex sets, and operator theory

Lecture I: Background. Rudiments of NC function theory

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The entire series is based on joint work with many collaborators, as well as the work of **many** others.

The model classification theorem

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In fact, if $\varphi : C(X) \to C(Y)$ is an isomorphism, then there exists a homeomorphism $h : Y \to X$ such that

$$\varphi(f) = f \circ h \ , \ f \in C(X).$$

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Thus $\varphi(f) = f \circ h$. (The converse is obvious.)

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- BTW — we are primarily interested in non-selfadjoint algebras (there will be also operator systems and C*-algebras later on).

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$$H^2_d = \left\{ f(z) = \sum_n a_n z^n : \langle f, f \rangle := \sum_n w_n |a_n|^2 < \infty \right\}$$

where $z = (z_1, \ldots, z_d)$, $n = (n_1, \ldots, n_d)$, and $z^n = z_1^{n_1} \cdots z_d^{n_d}$, and $w_n = \frac{n_1! \cdots n_d!}{(n_1 + \ldots n_d)!}$.

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Define $M_f(h) = fh$. Then $f \leftrightarrow M_f$ makes $\mathrm{Mult}\, H^2_d$ into an **operator algebra** with norm

$$\|f\|_{\mathrm{Mult}\, H^2_d} = \|M_f\|$$

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Problem B

Let $V, W \subset \mathbb{B}_d$ be two varieties. When are $\mathcal{M}_V$ and $\mathcal{M}_W$ isomorphic? Isometrically isomorphic? Completely Isometrically isomorphic? Similar/unitarily equivalent?

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(i) *$\mathcal{M}_V$ and $\mathcal{M}_W$ are (completely) isometrically isomorphic.*

(ii) *There exists a conformal automorphism $\alpha \in \mathrm{Aut}(\mathbb{B}_d)$ such that*

$$\alpha(W) = V.$$

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For homogeneous* varieties

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Theorem (Davidson-Ramsey-S, 2011; Hartz, 2012)

Let $V, W \subset \mathbb{B}_d$ be homogeneous varieties with $d < \infty$. Then the following are equivalent:

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(iii) *There is an invertible linear map on $\mathbb{C}^d$ which maps V onto W.*

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1. **What about the converse?**

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3. Is there a reasonable geometric structure that encodes the structure of the algebras $\mathcal{M}_V$?

Example II: Universal algebras for monomials ideals

Let $\mathcal{I} \triangleleft \mathbb{C}\langle z_1, \dots, z_d\rangle$ be an ideal generated by monomials.
Let $S^{\mathcal{I}} = (S_1^{\mathcal{I}}, \dots, S_d^{\mathcal{I}})$ be the **“universal" row contraction satisfying the relations in** $\mathcal{I}$.

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The operators $S_1^{\mathcal{I}}, \ldots, S_d^{\mathcal{I}}$, are the generators of Matsumoto's original subshift C*-algebras. The result says that the (**non-selfadjoint**) operator algebra they generate, encodes faithfully the combinatorial structure in the subshift.

A unifying framework

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. . . in a Banff meeting in 2015 it dawned on me that these and other algebras are all algebras of bounded **NC analytic functions** (and that this could be useful).

*** Photo by Brentkearney, taken from**
`https://commons.wikimedia.org/wiki/File:Banff_International_Research_Station.jpg` .

And now we begin . . .

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Example: the **nc unit ball** is $\mathfrak{B}_d := \left\{X \in \mathbb{M}^d : \|X\| < 1\right\}$, where

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Example: Let $U \subseteq \mathbb{C}$ and Ω the set of all matrices with spectrum in U. Then Ω is an open nc set (in $\mathbb{M}^1$) and every $f \in \mathcal{O}(U)$ is an nc holomorphic function on the set Ω (there is a very interesting converse).

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Theorem (KVV)

Let f be a nc function on Ω. For every $X \in \Omega_m$, $Y \in \Omega_n$, there exists a linear operator $\Delta f(X,Y)$ such that

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Viewing them as "function algebras" might be useful, suggests new ways of thinking about them and their quotients.

Noncommutative subvarieties of the nc unit ball

Definition (nc variety)

A **nc subvariety of the ball** (or simply a **variety**) is a set of the form

$$\mathfrak{V} = \{X \in \mathfrak{B}_d : f(X) = 0 \text{ for all } f \in S\}$$

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Theorem (Ball-Marx-Vinnikov/Agler-McCarthy)

Let $\mathfrak{V} \subseteq \mathfrak{B}_d$ be a variety. If $f : \mathfrak{V} \to \mathbb{M}^1$ is a bounded nc function, then there exists $F \in H^\infty(\mathfrak{B}_d)$ such that

$$f = F\big|_{\mathfrak{V}}$$

and

$$\sup_{X \in \mathfrak{B}_d} \|F(X)\| = \sup_{X \in \mathfrak{V}} \|f(X)\|$$

Algebras of bounded nc functions on varieties

Let $\mathfrak{V} \subseteq \mathfrak{B}_d$ be a nc variety. Define

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Motivating fact

The class of algebras $H^\infty(\mathfrak{V})$ (and $A(\mathfrak{V})$) contains many classes of operator algebras of interest.

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On the way ...

Develop function theory and complex geometry in the nc unit ball and on its varieties.

To be continued . . .

General references

(i) ***Foundations of Free Noncommutative Function Theory*** Kaliuzhnyi-Verbovetskyi and Vinnikov (The Bible*),

(ii) ***Aspects of Non-commutative Function Theory***, Agler and McCarthy (a readable survey).

***** - More precisely: *The Book of Leviticus*.