Residually finite-dimensional operator algebras

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Structure of operator algebras

A major tool of studying non-selfadjoint operator algebras is through the C*-algebras they generate. A **C*-cover** of an operator algebra $\mathcal{A}$ is a C*-algebra $\mathfrak{A}$ and a completely isometric homomorphism $\iota : \mathcal{A} \to \mathfrak{A}$ such that $C^*(\iota(\mathcal{A})) = \mathfrak{A}$.

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The two most significant C*-covers of $\mathcal{A}$ are the **maximal** and **minimal** C*-covers $(C^*_{\text{max}}(\mathcal{A}), \mu)$ and $(C^*_e(\mathcal{A}), \epsilon)$ (called the C*-envelope). They enjoy universal properties

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mu} & C^*_{\text{max}}(\mathcal{A}) \\ & \searrow^{\iota} & \downarrow \exists q \\ & & \mathfrak{A} \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\iota} & \mathfrak{A} \\ & \searrow^{\epsilon} & \downarrow \exists q \\ & & C^*_e(\mathcal{A}) \end{array}$$

Finite-dimensional operator algebras

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Example (Paulsen)

Let $\mathcal{S} = \mathrm{span}\{1, z\} \subset C(\mathbb{T})$ and define

$$\mathcal{A}_{\mathcal{S}} = \left\{ \begin{bmatrix} \lambda 1 & s \\ 0 & \mu 1 \end{bmatrix} : \lambda, \mu \in \mathbb{C}, s \in \mathcal{S} \right\}.$$

Thus, $\mathcal{A}_{\mathcal{S}}$ is finite-dimensional but it can be shown that $C_e^*(\mathcal{A}_{\mathcal{S}}) = M_2(C(\mathbb{T}))$.

Therefore, $\mathcal{A}_{\mathcal{S}}$ cannot be completely isometrically isomorphic to a subalgebra of a finite-dimensional C*-algebra.

Completely bounded structure

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Theorem (Clouâtre-R or is it older?)
Let $\mathcal{A} \subset B(\mathcal{H}_1), \mathcal{B} \subset B(\mathcal{H}_2)$ be unital operator algebras and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital completely bounded isomorphism. There exist completely isometric homomorphisms

$$\lambda : \mathcal{A} \to B(\mathcal{H}_1) \oplus B(\mathcal{H}_2), \quad \rho : \mathcal{B} \to B(\mathcal{H}_1) \oplus B(\mathcal{H}_2)$$

along with an invertible operator $Z \in B(\mathcal{H}_1) \oplus B(\mathcal{H}_2)$ such that

$$\Phi(a) = \rho^{-1}(Z\lambda(a)Z^{-1}), \quad a \in \mathcal{A}.$$

Similarity result

Theorem (Clouâtre-R, 2018)

An operator algebra $\mathcal{A}$ is finite-dimensional if and only if it is completely isometrically isomorphic to an operator algebra that is similar to an operator algebra that can be completely isometrically embedded in a finite direct sum of matrix algebras.

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This result relies on a proposition of Clouâtre and Marcoux that every finite-dimensional operator algebra has a very nice completely contractive homomorphism into a matrix algebra. It can be observed that their process leads to a completely bounded inverse as well.

RFD operator algebras

Definition (Clouâtre-Marcoux, Mittal-Paulsen)

An operator algebra is **residually finite-dimensional (RFD)** if there is a set of positive integers $\{r_\lambda : \lambda \in \Lambda\}$ and a completely isometric homomorphism

$$\rho : \mathcal{A} \to \prod_{\lambda \in \Lambda} M_{r_\lambda}.$$

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Proof.

(Partial sketch) For each element $a \in \mathcal{A}$ there is a state ψ such that $\psi(a^* a) = \|a\|^2$. Consider the associated GNS representation

$$\|\pi(a)\xi\|^2 = \langle \pi(a^* a)\xi, \xi \rangle = \psi(a^* a) = \|a\|^2.$$

Then $F = \pi(\mathcal{A})\xi$ is a finite-dimensional invariant subspace of $\pi(\mathcal{A})$ and so $P_F \pi(\cdot) P_F$ is a completely contractive homomorphism of $\mathcal{A}$ into a matrix algebra that achieves the norm of a. ∎

Examples and a non-example

The most immediate example of an RFD operator algebra is any subalgebra of an RFD C*-algebra. The more interesting cases are those RFD operator algebras that sit inside simple C*-algebras.

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3. *The tensor algebra of every directed graph.*

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3. *The tensor algebra of every directed graph.*

While the following operator algebra is not RFD:

$$\left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in B(\mathcal{H}) \right\}.$$

The C*-envelope: problems

There are finite-dimensional operator algebras whose C*-envelope is not RFD.

Example

For $d \geq 2$ let H^2_d be the Drury-Arveson space of holomorphic functions on the open unit ball $\mathbb{B}_d \subset \mathbb{C}^d$. Consider the unital subspace $\mathcal{S}$ given by the span of the multiplication operators $M_{z_1}, \dots, M_{z_d} \in B(H^2_d)$.

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$$\mathcal{A}_{\mathcal{S}} = \left\{ \begin{bmatrix} \lambda 1 & s \\ 0 & \mu 1 \end{bmatrix} : \lambda, \mu \in \mathbb{C}, s \in \mathcal{S} \right\}$$

which is finite-dimensional. However, it can be shown that $C^*_e(\mathcal{A}_{\mathcal{S}}) = M_2(\mathfrak{T}_d)$, where $\mathfrak{T}_d$ is the Toeplitz algebra. This C*-algebra cannot be RFD since it contains the compacts.

The C*-envelope: an RFD result

Theorem (Clouâtre-R, 2018)
Let $\mathcal{A} \subset \prod_{n=1}^{\infty} M_{r_n}$ be a unital operator algebra. Let

$$\mathfrak{K} = C^*(\mathcal{A}) \cap \bigoplus_{n=1}^{\infty} M_{r_n}$$

be the ideal of compact operators. If every C^*-algebra that is a quotient of $C^*(\mathcal{A})/\mathfrak{K}$ is RFD then $C_e^*(\mathcal{A})$ is RFD.

The maximal C*-cover

If $\mathcal{A}$ is not RFD then $C^*_{\max}(\mathcal{A})$ cannot be RFD since it contains $\mathcal{A}$. We are able to prove a partial converse of this.

Theorem (Clouâtre-R, 2018)

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Theorem (Clouâtre-R, 2018)

If $\mathcal{A}$ is a finite-dimensional operator algebra then $C^*_{\max}(\mathcal{A})$ is RFD.

Proof.

(Idea) We need to only find finite-dimensional Hilbert space representations that norm elements of the form

$$s = \sum_{i=1}^{n} s_{1,i} s_{2,i} \cdots s_{n_i,i}, \qquad s_{i,j} \in \mathcal{A} \cup \mathcal{A}^*.$$

The proof follows using the same GNS techniques as before and the universal property of the maximal C*-cover. ∎

Thanks!