Column-extreme free multipliers (and the free Smirnov class)

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Goal: Extend classical Hardy space/function theory results from one to several non-commuting (free) variables including characterizations of:

1. Extreme points of $[H^\infty(\mathbb{D})]_1$.
   - $H^\infty(\mathbb{D})$ = bounded analytic functions on $\mathbb{D} = (\mathbb{C})_1$.
2. The *Smirnov* class $\mathscr{N}^+$ of ratios b/a where $b, a \in H^\infty(\mathbb{D})$, and a is *outer*.
   - *i.e.* a is cyclic for S :=mult. by z, the *shift*, on $H^2(\mathbb{D})$.

$$H^2(\mathbb{D}) = \left\{ f(z) = \sum \hat{f}_n z^n \in \mathrm{Hol}(\mathbb{D}) \middle| \sum |\hat{f}_n|^2 < \infty \right\}.$$

3. Closed operators $T \sim H^\infty(\mathbb{D})$ (affiliated to the shift S).

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- Generally, $K : X \times X \to \mathbb{C}$ is a *positive kernel* if $\forall \, \{x_1, ..., x_N\} \subset X$ $0 \leq [K(x_j, x_k)]_{1 \leq j,k \leq N} \in \mathbb{C}^{N \times N}$.

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Hardy space: $H^2(\mathbb{D}) := \mathcal{H}(k)$ where $k : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is the *Szegö kernel*:

$$k(z,w) := \frac{1}{1 - zw^*}; \qquad z, w \in \mathbb{D}.$$

Multiplier algebras

Given a RKHS $\mathcal{H}(k)$ on X, the *multiplier algebra*, $\mathrm{Mult}(\mathcal{H}(k)) :=$ all $F : X \to \mathbb{C}$ s.t. $Fh \in \mathcal{H}(k)$ for any $h \in \mathcal{H}(k)$:

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- $b \in \mathscr{S}$ is *inner* if M_b is an isometry, and *outer* if M_b has dense range.

deBranges-Rovnyak spaces

Given contractive analytic $b \in \mathscr{S} = [H^\infty(\mathbb{D})]_1$, consider: $k^b : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$:

$$k^b(z, w) := \frac{1 - b(z)b(w)^*}{1 - zw^*}; \qquad \text{deBranges-Rovnyak kernel.}$$

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- One always has $S^* b \in \mathscr{H}(b)$ and $\|S^* b\|_{\mathscr{H}(b)} \leq 1 - |b(0)|^2$.

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2. b *is NOT column-extreme (CE): There is a non-zero* $a \in \mathscr{S}$ *so that the column* $\begin{bmatrix} b \\ a \end{bmatrix} \in \mathscr{S}(\mathbb{C}, \mathbb{C}^2) = [H^\infty(\mathbb{D}) \otimes \mathscr{L}(\mathbb{C}, \mathbb{C}^2)]_1$ *is Schur.*

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6. $\|S^* b\|_{\mathscr{H}(b)} < 1 - |b(0)|^2$. *(Sarason)*

The Smirnov class

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Goal: Extend these results to several non-commuting variables!

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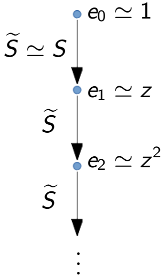
$\ell^2(\mathbb{N}_0) \simeq H^2(\mathbb{D})$:



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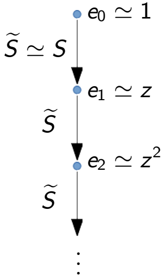

$\ell^2(\mathbb{F}^d)$: $\mathbb{F}^d$ = free monoid on d letters $\{1, ..., d\}$

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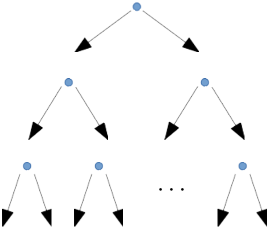

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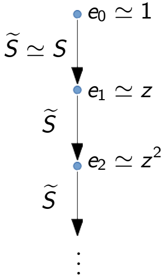

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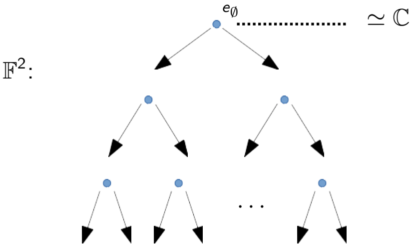

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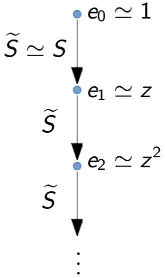

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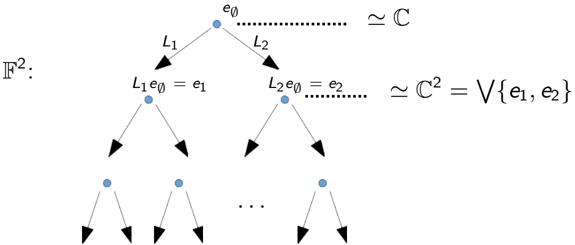

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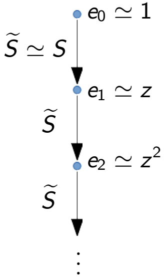

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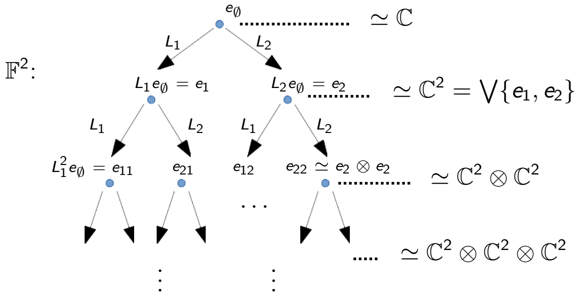

Left free shift: $L := (L_1, L_2, \ldots, L_d) : \ell^2(\mathbb{F}^d) \otimes \mathbb{C}^d \to \ell^2(\mathbb{F}^d)$

(row) isometry, $L_k^* L_j = \delta_{kj} I$

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Q1: Can $\ell^2(\mathbb{F}^d)$ be viewed as a Hardy-type space of several-variable "analytic functions" ? $\ell^2(\mathbb{F}^d) \simeq H^2(???)$??

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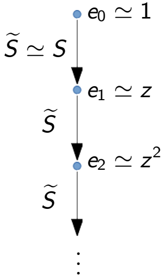

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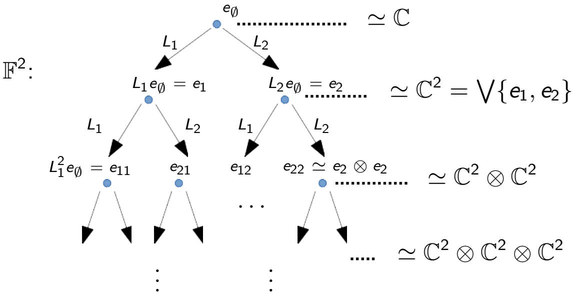

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Q2: Do Hardy space results have a faithful analogue when $d > 1$?

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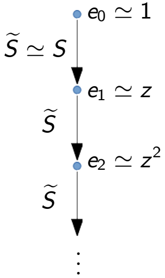

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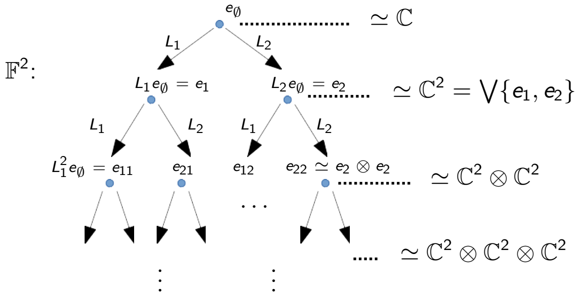

Left free shift: $L := (L_1, L_2, ..., L_d) : \ell^2(\mathbb{F}^d) \otimes \mathbb{C}^d \to \ell^2(\mathbb{F}^d)$

(row) isometry, $L_k^* L_j = \delta_{kj} I$

The Non-commutative open unit ball

Non-commutative (NC) open unit ball: $\mathbb{B}^d_{\mathbb{N}} := \bigsqcup_{n=1}^{\infty} \mathbb{B}^d_n$:

$\mathbb{B}^d_n := \left(\mathbb{C}^{n\times n} \otimes \mathbb{C}^d\right)_1$; strict row contractions on $\mathbb{C}^n$. $(\mathbb{B}^d_1 \simeq \mathbb{B}^d := (\mathbb{C}^d)_1)$

$Z \in \mathbb{B}^d_n$; $Z = (Z_1, ..., Z_d) : \mathbb{C}^n \otimes \mathbb{C}^d \to \mathbb{C}^n$; $Z_k \in \mathbb{C}^{n\times n}$.

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- Classical complex analysis extends naturally to this setting with purely algebraic proofs! [Popescu, K.-Verbovetskyi-Vinnikov, Agler-McCarthy]

Free Hardy space

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Note: $H^2(\mathbb{B}_\mathbb{N}^d) = \bigvee_{Z \in \mathbb{B}_\mathbb{N}^d} \mathrm{Ran}\,(K_Z)$: If $Z \in \mathbb{B}_n^d$, $K_Z^* : H^2(\mathbb{B}_\mathbb{N}^d) \to \left(\mathbb{C}^{n\times n}, \mathrm{tr}_n\right)$; *point evaluation*: $K_Z^* f := f(Z) \in \mathbb{C}^{n\times n}$.

Fock space and the free shifts

- *Fock space*, $F_d^2 :=$ direct sum of all tensor powers of $\mathbb{C}^d$ ($\simeq \ell^2(\mathbb{F}^d)$):

$$F_d^2 := \underbrace{\mathbb{C}}_{=\bigvee\{1\}} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d \otimes \mathbb{C}^d) \oplus (\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d) \oplus \cdots$$

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- Transpose unitary: $U_\dagger e_\alpha := e_{\alpha^\dagger}$, $U_\dagger L_k U_\dagger = R_k$.

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- Given NC polynomial $p \in \mathbb{C}\{L_1, ..., L_d\}$ define $\mathscr{U} : F_d^2 \to H^2(\mathbb{B}_{\mathbb{N}}^d)$:

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$$\boxed{(M_F^L f)(Z) = F(Z)f(Z), \quad (M_G^R f)(Z) = f(Z)G(Z)};$$

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Left and right free Schur classes: $\mathscr{L}_d := [L_d^\infty]_1; \quad \mathscr{R}_d := [R_d^\infty]_1.$

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Contractively contained in $F_d^2 = H^2(\mathbb{B}_{\mathbb{N}}^d)$ and co-invariant for the right/left free shifts, respectively (as before).

(Column) extreme points

Compare:

$d = 1$

Theorem

Given $b \in \mathscr{S} = [H^\infty(\mathbb{D})]_1$ TFAE:

1. *b is NOT column extreme:* $\exists 0 \neq a \in \mathscr{S}$ *so that* $\begin{bmatrix} b \\ a \end{bmatrix}$ *is Schur.*
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2. $b \in \mathscr{H}(b)$.
3. $\mathscr{H}(b)$ *is* $S-$*invariant.*
4. $\|S^* b\|_{\mathscr{H}(b)} < 1 - |b(0)|^2$.

$d \geq 1$

Theorem (J.-M.)

Given $B \in \mathscr{L}_d = [L_d^\infty]_1$ *TFAE:*

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(Column) extreme points

Compare:

$d = 1$

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Corollary (J.-M.)

If $B \in \mathscr{L}_d$ is column extreme, then it is an extreme point.

Applications to the Smirnov class

The *right free Smirnov class*:

$$\boxed{\mathscr{N}_d^+(R) := \left\{ F(Z) = A(Z)^{-1}B(Z) \in \mathrm{Hol}(\mathbb{B}_{\mathbb{N}}^d) \;\middle|\; B^\dagger, A^\dagger \in R_d^\infty,\ A^\dagger \text{ outer} \right\}}.$$

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A closed, densely-defined $T : \mathrm{Dom}\,(T) \subseteq F_d^2 \to F_d^2$ is *affiliated to the right free shift*, $T \sim R_d^\infty$, if $L_k \mathrm{Dom}\,(T) \subseteq \mathrm{Dom}\,(T)$ and $L_k T x = T L_k x$ for $x \in \mathrm{Dom}\,(T)$.

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Lemma (J.-M.)

If $F \in \mathscr{N}_d^+(R)$ then $T := M_F^R \sim R_d^\infty$ (defined on its maximal domain).

Right Affiliated = Right Smirnov

Theorem (J.-M.)

$T \sim R_d^\infty$ if and only if $T = M_F^R$ for some $F \in \mathcal{N}_d^+(R)$.

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Recall: Popescu-Wold decomposition $\Rightarrow$ $G(T) = \bigvee (L^\alpha \otimes I_2)\mathcal{W}(T)$ where $\mathcal{W}(T) = G(T) \ominus (L \otimes I_2)(G(T) \otimes \mathbb{C}^d)$ is the *wandering space*. $\dim \mathcal{W}(T) = \dim \mathcal{H}$.

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$$K^{G(T^*)}(Z,W) = K(Z,W) \otimes I_2 - K^{\mathrm{Ran}(J\Theta)}(Z,W) = \begin{pmatrix} K(Z,W) - K(Z,W)[B(Z)(\cdot) \otimes I_{\mathcal{H}} B(W)^*] & * \\ * & * \end{pmatrix},$$

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$$\begin{aligned}
0 &= \langle L\mathbf{x} \oplus TL\mathbf{x}, L_k g \oplus L_k TG \rangle && w \text{ is wandering} \\
&= \langle x_k, g \rangle + \langle L\mathbf{x}, T^* L_k Tg \rangle && \text{using the key assumption} \\
&= \langle x_k, g \rangle + \langle \mathbf{x}, (T^* \otimes I_d) L^* L_k Tg \rangle && \mathrm{Dom}\,(T^*) \text{ is } L^* \text{ invariant.} \\
&= \langle x_k, (I + T^*T) g \rangle. && \Rightarrow x_k \perp \mathrm{Ran}\,(I + T^*T) \Rightarrow x_k \equiv 0!!
\end{aligned}$$

Smirnov Factorization

Corollary (J.-M.)

Any $F \in \mathscr{N}_d^+(R)$ *can be written (uniquely) as* $F(Z) = A(Z)^{-1}B(Z)$ *where* $B^\dagger, A^\dagger \in \mathscr{R}_d$, *A is outer and the column* $\begin{bmatrix} A \\ B \end{bmatrix}$ *is inner.*

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Note: Any outer $B \in R_d^\infty$ or L_d^∞ is invertible on $\mathbb{B}_{\mathbb{N}}^d$. $\Rightarrow$ if $F(Z) = B(Z)^{-1}A(Z)$ with outer B then $F(Z) = 0 \Leftrightarrow A(Z) = 0$

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