Hilbert space operators with compatible off-diagonal corners

Laurent W. Marcoux

Department of Pure Mathematics
University of Waterloo

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This talk is joint work with:

- Leo Livshits (Colby College)
- Gordon MacDonald (U PEI)
- Heydar Radjavi (U Waterloo)

- $\mathcal{H}$ - a complex Hilbert space, separable.
- $\mathcal{B}(\mathcal{H})$ - bounded linear operators on $\mathcal{H}$. If $\mathcal{H} \simeq \mathbb{C}^n$, then $\mathcal{B}(\mathcal{H}) \simeq \mathbb{M}_n(\mathbb{C})$.
- $\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{B}(\mathcal{H}) : P = P^2 = P^*\}$ - orthogonal projections in $\mathcal{B}(\mathcal{H})$.

Let $T \in \mathcal{B}(\mathcal{H})$, $P \in \mathcal{P}(\mathcal{H})$. We refer to $(I - P)TP$ as an "off-diagonal corner" of T. Clearly $PT(I - P)$ is again an off-diagonal corner of T.

If

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

then the off-diagonal corners are T_2 and T_3.

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The invariant subspace problem

Let $\mathcal{H}$ be an infinite-dimensional, separable, complex Hilbert space, and $T \in \mathcal{B}(\mathcal{H})$. Does there exist $P \in \mathcal{P}(\mathcal{H})$ with $0 \neq P \neq I$ such that

$$T = (I - P)TP?$$

The reductive operator conjecture

Suppose $T \in \mathcal{B}(\mathcal{H})$ satisfies the following property:
if $P \in \mathcal{P}(\mathcal{H})$ and $(I - P)TP = 0$, then $PT(I - P) = 0$.
(We say that T is (orthogonally) reductive.) Then T is normal.

Theorem. *(Dyer-Pederson-Porcelli – 1972) The invariant subspace problem has an affirmative answer if and only if the reductive operator conjecture is true.*

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- We say that an operator $T \in \mathcal{B}(\mathcal{H})$ has property (CR) – the common rank property – if for all $P \in \mathcal{P}(\mathcal{H})$,

$$\operatorname{rank}(I-P)TP = \operatorname{rank} PT(I-P).$$

- We say that an operator $T \in \mathcal{B}(\mathcal{H})$ has property (CN) – the common norm property – if for all $P \in \mathcal{P}(\mathcal{H})$,

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Proposition. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ and that T has (CN).*

(a) *For all $\lambda, \mu \in \mathbb{C}$, we have that $\lambda I + \mu T$ and T^* have (CN).*

(b) *Suppose that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. If there exist $A \in \mathcal{B}(\mathcal{H}_1)$ and $D \in \mathcal{B}(\mathcal{H}_2)$ such that $T = A \oplus D$, then A and D both have (CN).*

(c) *If $V \in \mathcal{B}(\mathcal{H})$ is unitary, then $V^* T V$ has (CN).*

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In fact, for (CN), we can do a bit better:

Proposition.

- *The set $\mathfrak{G}_{\mathfrak{norm}}$ of operators with (CN) is closed.*
- *If $T \in \mathcal{B}(\mathcal{H})$ has (CN) and there exists $S \in \overline{\mathcal{U}(T)}$ of the form $S = A \oplus D$, then A, D have (CN).*

Unitaries play a special role in our investigations: suppose $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is unitary. Then $U^*U = I = UU^*$ implies that

$$BB^* = I - AA^*, \qquad C^*C = I - A^*A.$$

Since the norm of B (resp. the norm of C) is determined by $\sigma(BB^*)$ (resp. $\sigma(C^*C)$), $\|B\| \neq \|C\|$ implies that either

- $0 \in \sigma(AA^*)$ but $0 \notin \sigma(A^*A)$ or
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Proposition. *Let $n \geq 2$.*

- *If $T \in \mathbb{M}_n(\mathbb{C})$ has (CN) or (CR), then T is normal. (This is elementary.)*
- *If $U \in \mathbb{M}_n(\mathbb{C})$ is unitary, then U has both (CN) and (CR).*

Corollary. *Let $n \geq 2$. If $T \in \mathbb{M}_n(\mathbb{C})$ is either hermitian or unitary, then for all $\lambda, \mu \in \mathbb{C}$,*

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In both cases, the spectrum of the operator lies on a "circline", in the sense of Möbius maps and complex analysis.

Proposition. *If $T \in \mathbb{M}_4(\mathbb{C})$ has property (CN) or (CR), then $\mu(T)$ has the same property for any Möbius map μ that is finite on the spectrum of T.* •

Theorem.

- *If $n \in \{2, 3\}$, then T has (CN) if and only if T has (CR) if and only if T is normal.*
- *If $n \geq 4$, then T has (CN) if and only if T has (CR), and this happens if and only if T is normal with circlinear spectrum.*

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The above characterization of (CN) (and of (CR)) fails in the infinite-dimensional setting.

Let U denote the bilateral shift operator $Ue_n = e_{n-1}$ on $\ell_2(\mathbb{Z})$. Then U is normal, $\sigma(U) = \mathbb{T}$, and

$$U \simeq \begin{bmatrix} S & U_2 \\ 0 & S^* \end{bmatrix}$$

where S is the unilateral forward shift, and where U_2 has norm one and rank one.

An operator is said to be strongly reductive if

$$\lim_n \|(I - P_n)TP_n\| = 0 \quad \text{implies} \quad \lim_n \|P_n T(I - P_n)\| = 0.$$

Clearly (CN) implies strongly reductive.

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Clearly (CN) implies strongly reductive.

Corollary. *If $T \in \mathcal{B}(\mathcal{H})$ has (CN), then T is normal and has Lavrentiev spectrum (no interior and does not disconnect the plane).*

Proof.

- Harrison (1975) showed that T strongly reductive implies $\sigma(T)$ is Lavrentiev, and
- Apostol, Foiaș, and Voiculescu (1976) showed that T strongly reductive implies that T is normal.

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Note that the spectrum of T must also be circlinear. If $\alpha, \beta, \gamma, \delta \in \sigma(T)$, then $T \simeq_a T_0 \oplus \mathrm{diag}(\alpha, \beta, \gamma, \delta)$, and the latter must have (CN).

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Definition. *For $T \in \mathcal{B}(\mathcal{H})$, the numerical range of T is*

$$W(T) = \{\langle Te, e \rangle, \quad e \in \mathcal{H}, \|e\| = 1\}.$$

The essential numerical range is

$$W_e(T) = \{\varphi(\pi(T)) : \varphi \text{ is a state on } \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})\}.$$

Theorem. *(Fillmore-Stampfli-Williams–1972) For $T \in \mathcal{B}(\mathcal{H})$, TFAE*

- $0 \in W_e(T)$;
- *There exists an orthonormal sequence $(e_n)_n$ such that $\lim_n \langle Te_n, e_n \rangle = 0$.*

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Definition. *Let $\mathcal{K} \subseteq \mathcal{H}$, $R \in \mathcal{B}(\mathcal{K})$. Then $T \in \mathcal{B}(\mathcal{H})$ is said to be a dilation of R if there exist B, C, D such that*

$$T = \begin{bmatrix} R & B \\ C & D \end{bmatrix}.$$

Theorem. *(Choi-Li–2001) Suppose that $A \in \mathcal{B}(\mathcal{H})$, and $T \in \mathbb{M}_3(\mathbb{C})$ has a non-trivial reducing subspace. Then A has a dilation that is unitarily equivalent to $T \otimes I$ if and only if $W(A) \subseteq W(T)$.*

Theorem. *If $V \in \mathbb{M}_3(\mathbb{C})$ is unitary and 0 lies in the interior of $W(V)$, then $V \otimes I$ does not have (CN).*

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Theorem. *If $V \in \mathbb{M}_3(\mathbb{C})$ is unitary and 0 lies in the interior of $W(V)$, then $V \otimes I$ does not have (CN).* •

For unitary operators, this is the only obstruction.

Corollary. *The following are equivalent for a unitary operator $U \in \mathcal{B}(\mathcal{H})$.*

(a) *U has (CN).*

(b) *0 does not lie in the interior of $W_e(U)$.*

(c) *There exists a half-circle $\mathcal{C}$ of $\mathbb{T}$ such that $\sigma_e(U) \subseteq \mathcal{C}$.*

Proof. Suppose U unitary does not have (CN), say $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $\|B\| \neq \|C\|$. Recall that this means $0 \in \sigma(AA^*)$ but $0 \notin \sigma(A^*A)$ or vice-versa. Show that A is semi-Fredholm with non-zero index, and use Fillmore-Stampfli-Williams to conclude that $0 \in W_e(A) \subseteq W_e(U)$. □

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(c) *There exists a half-circle* $\mathcal{C}$ *of* $\mathbb{T}$ *such that* $\sigma_e(U) \subseteq \mathcal{C}$.

Proof. Suppose U unitary does not have (CN), say $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $\|B\| \neq \|C\|$. Recall that this means $0 \in \sigma(AA^*)$ but $0 \notin \sigma(A^*A)$ or vice-versa. Show that A is semi-Fredholm with non-zero index, and use Fillmore-Stampfli-Williams to conclude that $0 \in W_e(A) \subseteq W_e(U)$.

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For unitary operators, this is the only obstruction.

Corollary. *The following are equivalent for a unitary operator $U \in \mathcal{B}(\mathcal{H})$.*

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Theorem. *Let $T \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalent.*

(a) *T has (CN).*

(b) *One of the following holds.*

   (i) *There exist $\lambda, \mu \in \mathbb{C}$ and $L = L^* \in \mathcal{B}(\mathcal{H})$ such that $T = \lambda I + \mu L$.*

   (ii) *There exist $\lambda, \mu \in \mathbb{C}$ with $\mu \neq 0$ and a unitary operator $U \in \mathcal{B}(\mathcal{H})$ with $\sigma_e(U) \subseteq \mathbb{T} \cap \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ such that $T = \lambda I + \mu U$.*

Property (CR) is more subtle than property (CN). As we shall see, there exist two approximately unitarily equivalent unitary operators U and V such that V has property (CR) but U does not!

Proposition. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ has (CR). Then T is biquasitriangular; that is,* $\text{ind}\,(T - \lambda I) = 0$ *for all* $\lambda \in \rho_{sF}(T)$.

Proof. Suppose that $\text{nul}\,(T - \lambda I)^* \leq \text{nul}\,(T - \lambda I)$ for some λ. Write $\mathcal{H} = \ker(T - \lambda I) \oplus (\ker(T - \lambda I))^{\perp}$, and write

$$(T - \lambda I) = \begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix}.$$

By (CR) we see that $B = 0$ and thus $\text{nul}\,(T - \lambda I)^* \geq \text{nul}\,(T - \lambda I)$.

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Elementary observation

- The eigenvalues of any $T \in \mathcal{B}(\mathcal{H})$ with (CR) are reducing eigenvalues, and they are either co-linear or co-circular.
- Thus $T \simeq T_0 \oplus M$, where T_0 has no eigenvalues, and M is a normal operator which has cocircular spectrum.

Proposition. *Let $T \in \mathcal{B}(\mathcal{H})$ and suppose that T has (CR). Then there exist α, β, γ and $\delta \in \mathbb{C}$, not all equal to zero, and an operator $F \in \mathcal{B}(\mathcal{H})$ of rank at most three such that*

$$\alpha I + \beta T + \gamma T^* + \delta T^* T + F = 0.$$

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Proof. Fix $0 \neq \xi \in \mathcal{H}$. We first claim that the set

$$S_\xi = \{\xi, T\xi, T^*\xi, T^*T\xi\}$$

is linearly dependent.

Let $\mathcal{M}_\xi = \mathrm{span}\,\{\xi, T\xi\}$. Note that $T\xi \in \mathcal{M}_\xi$ implies that $\mathrm{rank}\, P_\xi^\perp T P_\xi \in \{0, 1\}$. (CR) implies $\mathrm{rank}\, P_\xi^\perp T^* P_\xi \in \{0, 1\}$.

As $0 \neq \xi \in \mathcal{H}$ was arbitrary, we see that the set $\{I, T, T^*, T^*T\}$ is *locally linearly dependent*. By a result of Aupetit – 1988, there exist α, β, γ, and $\delta \in \mathbb{C}$, not all equal to zero, such that

$$\mathrm{rank}\,(\alpha I + \beta T + \gamma T^* + \delta T^*T) \leq 3.$$

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Theorem. *Suppose* $\alpha, \beta, \gamma,$ *and* $\delta \in \mathbb{C}$, *not all equal to zero, and*

$$\mathrm{rank}\,(\alpha I + \beta T + \gamma T^* + \delta T^* T) \leq 3.$$

(a) *If* $\delta = 0$, *there exists* $R = R^*$, $L \in \mathcal{B}(\mathcal{H})$ *with* $\mathrm{rank} \leq 6$, *and* $\mu, \lambda \in \mathbb{C}$ *such that*

$$T = \lambda(R + L) + \mu I.$$

(b) *If* $\delta \neq 0$, *there exists a unitary operator* V *and* L, μ, λ *as above such that*

$$T = \lambda(V + L) + \mu I.$$

With the help of a long technical lemma:

Theorem. *Let $\mathcal{H}$ be an infinite-dimensional, complex Hilbert space, and let $T \in \mathcal{B}(\mathcal{H})$. If T satisfies (CR), then there exist $\lambda, \mu \in \mathbb{C}$ and $A \in \mathcal{B}(\mathcal{H})$ with A either selfadjoint or an orthogonally reductive unitary operator such that $T = \lambda A + \mu I$.*

Question. Is the converse true?

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Theorem. *(Wermer–1952) U a unitary operator. TFAE*

- *U fails to be reductive.*
- *Lebesgue measure is absolutely continuous with respect to the spectral measure μ for U.*

Since any operator with (CR) is necessarily reductive, this provides a measure-theoretic obstruction to (CR) for unitary operators.

Proposition. *Let $U \in \mathcal{B}(\mathcal{H})$ be unitary.*

- *If $\sigma(U) \neq \mathbb{T}$. Then U has (CR).*
- *If $(d_n)_n$ is a sequence in $\mathbb{T}$ and $U = \mathrm{diag}(d_n)_n$, then U has (CR).*
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THANK YOU FOR YOUR ATTENTION.