Multivariate pseudospectrum and topological physics

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COSy

Spectrum and pseudospectrum for two Hermitian matrices

Physics observables are Hermitian, so we prefer to work with n-tuples of Hermitian matrices (operators, C^*-algebra elements).

Definition

Given A and B Hermitian, define their joint spectrum $\Lambda(A, B)$ as a subset of $\mathbb{R}^2$ via

$$(\lambda_1, \lambda_2) \in \Lambda(A, B) \quad \iff \quad \lambda_1 + i\lambda_2 \in \sigma(A + iB)$$

When $AB \approx BA$ one can show

$$(\lambda_1, \lambda_2) \in \Lambda(A, B) \quad \implies \exists \mathbf{v} \text{ unit vector } A\mathbf{v} \approx \lambda_1 \mathbf{v} \ \& \ B\mathbf{v} \approx \lambda_2 \mathbf{v}$$

Characteristic polynomial

In the matrix case, the characteristic polynomial of (A, B) is essentially $\mathrm{char}(A + iB)$:

$$(\lambda_1, \lambda_2) \in \Lambda(A, B) \quad \Longleftrightarrow \quad \det\left((A - \lambda_1) + i\,(B - \lambda_2)\right) = 0$$

Since this is a polynomial in $z = \lambda_1 + i\lambda_2$ we see, as expected:

$\Lambda(A, B)$ is always a finite set when A and B are finite matrices.

More meaningful is the joint pseudospectrum, essentially just the pseudospectrum of $A + iB$:

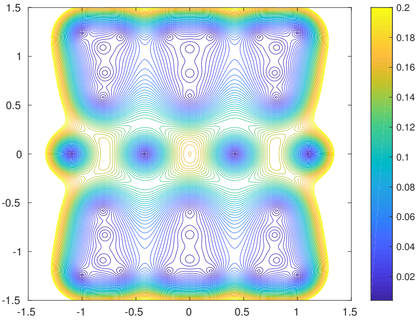
Definition

Given A and B Hermitian, define $\Lambda_\epsilon(A, B)$ a the subset of $\mathbb{R}^2$ via

$$(\lambda_1, \lambda_2) \in \Lambda_\epsilon(A, B) \quad \Longleftrightarrow \quad \left\| \left((A - \lambda_1) + i\,(B - \lambda_2)\right)^{-1} \right\|^{-1} \le \epsilon$$

Model of polyacetylene, with defects

Here is $\Lambda_\epsilon(X,H)$ for X position and H the Hamiltonian, in a model of polyacetylene. The unpaired zero-modes (and the K-theory of graded C^*-algebras) can explain how polyacetylene can be conducting.



$$\left\|\left((X-\lambda_1)+i(H-\lambda_2)\right)^{-1}\right\|^{-1}$$

W. P. Su, J. R. Schrieffer, and A. J. Heeger. "Solitons in polyacetylene." Physical Review Letters 42.25 (1979): 1698.

Generalizing to three Hermitian matrices

Notice

$$\left(\left(A-\lambda_1\right)+i\left(B-\lambda_2\right)\right) \text{ is singular}$$
$$\Leftrightarrow \begin{bmatrix} 0 & \left(A-\lambda_1\right)-i\left(B-\lambda_2\right) \\ \left(A-\lambda_1\right)+i\left(B-\lambda_2\right) & 0 \end{bmatrix} \text{ is singular}$$

and

$$\left\|\left(\left(A-\lambda_1\right)+i\left(B-\lambda_2\right)\right)^{-1}\right\|^{-1}$$
$$=\lambda_{\min}\begin{bmatrix} 0 & \left(A-\lambda_1\right)-i\left(B-\lambda_2\right) \\ \left(A-\lambda_1\right)+i\left(B-\lambda_2\right) & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & \left(A-\lambda_1\right)-i\left(B-\lambda_2\right) \\ \left(A-\lambda_1\right)+i\left(B-\lambda_2\right) & 0 \end{bmatrix}$$
$$=\left(A-\lambda_1\right)\otimes\sigma_1+\left(B-\lambda_2\right)\otimes\sigma_2$$

Generalizing to three Hermitian matrices

We extend, using the third Pauli spin matrix:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

so work with

$$\begin{aligned} L_{\boldsymbol{\lambda}}(X_1, X_2, X_3) &= \sum (X_j - \lambda_j) \otimes \sigma_j \\ &= \begin{bmatrix} X_3 & X_1 - iX_2 \\ X_1 + iX_2 & -X_3 \end{bmatrix} - \begin{bmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{bmatrix} \end{aligned}$$

Definition

Given X_1, X_2 and X_3 Hermitian, define $\Lambda_\epsilon(X_1, X_2, X_3)$ a the subset of $\mathbb{R}^3$ via

$$\boldsymbol{\lambda} \in \Lambda_\epsilon(X_1, X_2, X_3) \quad \Longleftrightarrow \quad \lambda_{\min}\left(L_{\boldsymbol{\lambda}}(X_1, X_2, X_3)\right) \leq \epsilon$$

and especially

$$\boldsymbol{\lambda} \in \Lambda(X_1, X_2, X_3) \quad \Longleftrightarrow \quad L_{\boldsymbol{\lambda}}(X_1, X_2, X_3) \text{ is singular}$$

Characteristic polynomial

$$(\lambda_1, \lambda_2, \lambda_2) \in \Lambda(X_1, X_2, X_3) \quad \iff \quad \det\left(L_{\boldsymbol{\lambda}}(X_1, X_2, X_3)\right) = 0$$

Since this characteristic polynomial will generally fail to be related to any analytic function, *the joint spectrum can be infinite.* However:

$\Lambda(X_1, X_2, X_3)$ is finite for *commuting* finite (Hermitian) matrices.

Finite matrices, infinite spectrum

Theorem (Kisel)

For $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$\Lambda(X, Y, Z) = S^2.$$

Proof.

$$\begin{bmatrix} Z & X - iY \\ X + iY & -Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 - z & 0 & -x + iy & 0 \\ 0 & -1 - z & 2 & -x + iy \\ -x - iy & 2 & -1 + z & 0 \\ 0 & -x - iy & 0 & 1 + z \end{bmatrix} = (x^2 + y^2 + z^2 + 1)^2 - 4 \quad \square$$

Kisil, Vladimir. "Möbius transformations and monogenic functional calculus." Electronic Research Announcements of the American Mathematical Society 2.1 (1996): 26-33.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{10} & -i \\ i & \frac{1}{5} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & \frac{1}{10} \\ \mathbf{\frac{1}{10}} & -\frac{3}{5} \end{bmatrix}$$

leads to characteristic polynomial

$$x^4 + 2x^2y^2 - \frac{3x^2y}{5} + 2x^2z^2 - \frac{4x^2z}{5} + \frac{57x^2}{20} + \frac{xy}{5} - \frac{84xz}{25} + \frac{321x}{500} + y^4$$
$$- \frac{3y^3}{5} + 2y^2z^2 - \frac{4y^2z}{5} + \frac{9y^2}{100} - \frac{3yz^2}{5} + \frac{14yz}{25} - \frac{8y}{125} + z^4 - \frac{4z^3}{5} + \frac{157z^2}{100} - \frac{153z}{250} + \frac{53}{2000}$$

and the joint spectrum (zero locus) is this surface:

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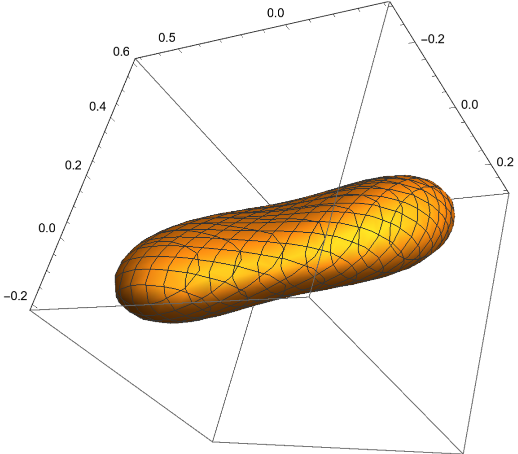

$$A = \begin{bmatrix} 0 & \frac{11}{10} & 0 & \frac{1}{10} \\ \frac{11}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{10} & 0 & 1 & \frac{3}{10} \end{bmatrix}, B = \begin{bmatrix} 0 & -\frac{i}{2} & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, C = \begin{bmatrix} \frac{11}{10} & 0 & 0 & 0 \\ 0 & -\frac{11}{10} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

has joint spectrum whose top part looks like:

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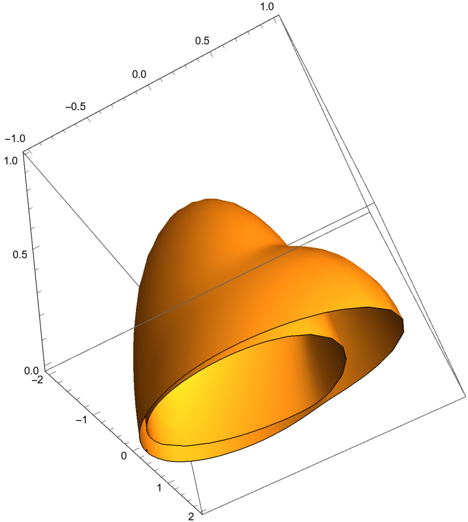

Labeling components of the resolvent

Every point $\boldsymbol{\lambda}$ not in $\Lambda(X_1, X_2, X_3)$ has an associated value in $\mathbb{Z} = K_0(\mathbb{C})$ given by

$$\mathrm{ind}_{\boldsymbol{\lambda}}(X_1, X_2, X_3) = \frac{1}{2}\mathrm{sig}\left(L_{\boldsymbol{\lambda}}(X_1, X_2, X_3)\right).$$

This index can be shown to equal the Chern number of a 2D infinite system (noninteracting fermions) when X_1 and X_2 are position observables for a large finite portion of the system and X_3 is the truncated Hamiltonian.

Some string theorist also study this index, in the context of the K-theory of D-branes. The index as given here is only for a D-brane in dimension 3, so with three almost commuting position observables.

L. and Schulz-Baldes, Finite volume calculation of K-theory invariants, N. Y. J. Math., 2017.

Berenstein and Dzienkowski. Matrix embeddings on flat $\mathbb{R}^3$ and the geometry of membranes. Physical Review D, 86(8):086001, 2012.