Regular dilation and Nica-covariant representation on right LCM semigroups

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Question

Given a contractive representation T of a semigroup P on $\mathcal{B}(\mathcal{H})$. When can we find an "isometric dilation" V in the sense that there exists a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ so that for all $p \in P$,

$$P_{\mathcal{H}} V(p)|_{\mathcal{H}} = T(p).$$

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2. *Ando extended Sz.Nagy's theorem to $\mathbb{N}^2$.*
3. *However, there exists a contractive representation of $\mathbb{N}^3$ that fails to have isometric dilation.*

Question

Any other type of dilation?

Definition

An isometric dilation (W_i) *for* (T_i) *is called* ∗*-regular if it has an additional property that for any* $n \in \mathbb{Z}^k$,

$$T^{n^+} T^{*n^-} = P_{\mathcal{H}} W^{*n^-} W^{n^+}\big|_{\mathcal{H}}.$$

Here, $n^+ = \max\{n, 0\}$ *and* $n^- = \max\{-n, 0\}$.

Theorem (Brehmer 1961)

A family of commuting contractions $T_1, \cdots, T_k$ *(i.e. a contractive representation of* $\mathbb{N}^k$*) has a* ∗*-regular dilation if and only if for any* $J \subseteq \{1, 2, \cdots, k\}$,

$$\sum_{U \subseteq J} (-1)^{|U|} T(e_U) T(e_U)^* \geq 0.$$

Example

For $k=2$, let T_1, T_2 be two commuting contractions. Brehmer's condition is equivalent to

$$I - T_1 T_1^* - T_2 T_2^* + T_1 T_2 (T_1 T_2)^* \geq 0.$$

This condition is stronger than Ando's dilation, and ∗-regular dilation is also stronger than isometric dilation.

Our study focus on ∗-regular dilation on semigroups.

Example

Given a lattice ordered group (G, P)*, every element* $g \in G$ *has a unique decomposition* $g = g_-^{-1} g_+$*. In this case, it is fairly straightforward to extend regular dilation by saying* $V : P \to \mathcal{B}(\mathcal{K})$ *is a* $*$*-regular dilation for* T *if for all* $g \in G$*,*

$$T(g_+)T^*(g_-) = P_{\mathcal{H}} V^*(g_-) V(g_+)\big|_{\mathcal{H}}.$$

However, lattice order condition is very restrictive. It does not contain many interesting examples of semigroups (free semigroup, Artin-type monoids, etc.). Nevertheless, we were able to characterize ∗-regular dilation on graph product of $\mathbb{N}$, which is an important class of quasi-lattice ordered semigroup.

Theorem (L. 2017)

Let T be a contractive representation of graph product of $\mathbb{N}$. Then the following are equivalent:

1. *T has a ∗-regular dilation;*
2. *T has a minimal isometric Nica-covariant dilation;*
3. *For every finite $W \subset V$,*

$$\sum_{\substack{U \subseteq W \\ U \text{ is a clique}}} (-1)^{|U|} T_U T_U^* \geq 0.$$

Here, $T_U = \prod_{v \in U} T_v$.

Definition

Given a unital semigroup P inside a group G, P defines a partial order $\leq$ by $x \leq y$ when $x^{-1}y \in P$. (G, P) is called quasi-lattice ordered if for any finite subset $F \subset G$ with an upper bound, F has a least upper bound. We denote the least upper bound by $\vee F$.

Definition

Given a quasi-lattice ordered group (G, P), an isometric representation $V : P \to \mathcal{B}(\mathcal{H})$ is called isometric Nica-covariant if for any $p, q \in P$,

$$V(p)V(p)^* V(q)V(q)^* = \begin{cases} V(p \vee q)V(p \vee q)^*, p \vee q \neq \infty \\ 0, p \vee q = \infty \end{cases}$$

Definition

A semigroup P is called a right LCM semigroup if it's left-cancellative and for any $p, q \in P$, either $pP \cap qP = \emptyset$ or $pP \cap qP = rP$ for some $r \in P$. Here, r may not be unique. We denote

$$p \vee q = \{r : rP = pP \cap qP\}.$$

It is clear that when $r, s \in p \vee q$, $r = su$ for some unit u.

Definition

Given a right LCM semigroup P, an isometric representation $V : P \to \mathcal{B}(\mathcal{H})$ is called isometric Nica-covariant if for any $p, q \in P$,

$$V(p)V(p)^*V(q)V(q)^* = \begin{cases} V(r)V(r)^*, r \in p \vee q \neq \emptyset \\ 0, p \vee q = \emptyset \end{cases}$$

We can 'reverse engineer' a definition of $*$-regular dilation from Nica-covariance.

Definition

We say a contractive representation T of a right LCM semigroup is $*$-regular if it has an isometric dilation V so that for all $p, q \in P$,

$$T(p^{-1}r)T(q^{-1}r)^* = P_{\mathcal{H}}V(p)^*V(q)|_{\mathcal{H}}.$$

Here, by convention, when $p \vee q = \emptyset$,

$$0 = P_{\mathcal{H}}V(p)^*V(q)|_{\mathcal{H}}.$$

Theorem (L.)

The following are equivalent

1. T *is* $*$*-regular*
2. T *has a minimal isometric Nica-covariant dilation*
3. *For any finite* $F \subset P$*, pick* $s_U \in \vee U$ *for each* $U \subset F$*, we have*
$$Z(F) = \sum_{U \subset F} (-1)^{|U|} T(s_U) T(s_U)^* \geq 0.$$

Lemma

Suppose $F = \{p_1 a, p_2, \cdots, p_n\}$. *Denote*

$$\begin{aligned} F_1 &= \{p_1, p_2, \cdots, p_n\} \\ F_2 &= \{a, p_1^{-1}(p_1 \vee p_2), \cdots, p_1^{-1}(p_1 \vee p_n)\} \end{aligned}$$

Then $Z(F) = Z(F_1) + T(p_1) Z(F_2) T(p_1)^*$. *Moreover,* $Z(F) \geq 0$ *if* $Z(F_1), Z(F_2) \geq 0$.

Suppose P has descending chain condition for $\leq_r$. Let P_{min} be the set of minimal elements. Suppose a set P_0 satisfies

1. $P_{min} \subset P_0$
2. For all $x \in P_{min}$ and $y \in P_0$ with $x \vee y \neq \infty$, $x^{-1}(x \vee y) \in P_0$.

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Moreover, if P *is an Ore semigroup, under 'some extra assumption', these are equivalent to*

- $Z(F) \geq 0$ *for all finite* $F \subset P_{min}$.

Example

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When $P=\mathbb{F}_n^+$, our theorem states T is $*$-regular if and only if

$$I-\sum_{i=1}^n T(e_i)T(e_i)^*\geq 0.$$

This is no other than the well-known Frazho-Bunce-Popescu dilation of row contractions into row isometries.

Our result applies to a number of ‘nicely-generated’ semigroups:

1. The Thompson's $F^+ = \langle x_0, x_1, \cdots | x_n x_k = x_k x_{n+1}, k < n \rangle$.
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2. The Baumslag-Solitar monoid $B_{c,d}$ which is the monoid generated by a, b with the relation $ab^c = b^d a$. We can take $P_0 = \{b, a, ba, \cdots, b^{d-1}a\}$.

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3. The semigroup $\mathbb{N} \rtimes \mathbb{N}^\times$ where

$$(x, a)(y, b) = (x + qy, ab).$$

We can take $P_0 = \{(1, 1), (i, p) : p \text{ prime}, 0 \leq i < p\}$.

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4. For all finite type Artin monoids A_M^+, it suffices to check P_{min} which is the set of generators.

Example

$B_3^+ = \langle e_1, e_2 : e_1 e_2 e_1 = e_2 e_1 e_2 \rangle$. *$T$ is $*$-regular if and only if*

$$I - T_1 T_1^* - T_2 T_2^* + T_1 T_2 T_1 T_1^* T_2^* T_1^* \geq 0.$$

Definition

Let $\Gamma = (V, E)$ be a countable simple undirected graph. Suppose $P = (P_v)_{v \in V}$ is a countable collection of right LCM semigroups. The graph product $\Gamma_{v \in V} P_v$ is the semigroup defined by taking the free product $*_{v \in V} P_v$ modulo the relation $p \in P_v$ commutes with $q \in P_u$ whenever (u, v) is an edge in the graph Γ.

Theorem (Fountain, Kambites, 2009)

Let P_v be a collection of right LCM semigroups. Then their graph product $\Gamma_{v \in V} P_v$ is also right LCM.

Theorem (L.)

Let P_Γ be a graph product of a collection of right LCM smeigroups $(P_v)_{v\in V}$, and $T: P_\Gamma \to \mathcal{B}(\mathcal{H})$ be a contractive representation. Then the following are equivalent:

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1. *For every finite set $F \subset P_\Gamma$, $Z(F) \geq 0$.*
2. *For every finite set $e \notin F \subset \bigcup_{v\in V} P_v$, $Z(F) \geq 0$.*

Example

If we take $P_v = \mathbb{N}$ for all v, it is clear that P_v has the descending chain property and the only minimal element is its generator e_v. This recovers our earlier result of $*$-regular dilation on graph products of $\mathbb{N}$.

Example

If the graph is a complete graph, the graph product becomes a direct product. We say two representation T_v, T_u $*$-commute if $T_v(p)$ $*$-commute with each $T_u(q)$ for all $p \in P_v, q \in P_u$ $(u \neq v)$. As a corollary of our main result, graph product of $*$-commuting $*$-regular representation of right LCM semigroups are also $*$-regular.

Thank you