Complexity and capacity of quantum channels

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Overview

1. Classical channels
   Confusability graphs and zero-error capacity (Shannon)

2. Operator systems from quantum channels
   Non-commutative confusability graphs (Duan-Severini-Winter)
   New parameters for operator systems:
   quantum complexity and quantum subcomplexity

3. Application: estimating quantum zero-error capacity

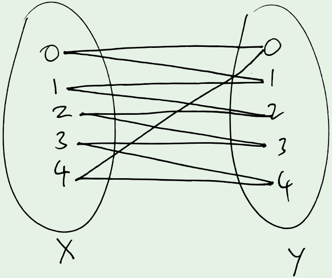
1. Classical channels

Classical channels

- A noisy channel $N\colon X \rightsquigarrow Y$ with input alphabet X, output alphabet Y is a map $N\colon X \to \{\text{non-empty subsets of } Y\}$
- View $N(x)$ as the set of possible outputs of N, given input x
- Equivalently: N is an (X, Y) bipartite graph for which every $x \in X$ emits at least one edge.

Example

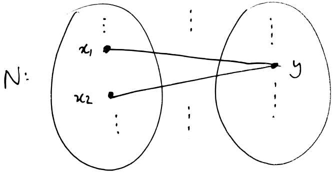
Let $N_5\colon \mathbb{Z}_5 \rightsquigarrow \mathbb{Z}_5$, $N(x) = \{x, x+1\}$ for $x \in \mathbb{Z}_5$.



The confusability graph and complexity

Let $N\colon X \rightsquigarrow Y$ be a channel.
Say $x_1, x_2 \in X$ with $x_1 \neq x_2$ are *confusable* for N if

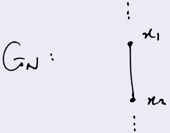
$$N(x_1) \cap N(x_2) \neq \emptyset.$$



Definition

The confusability graph G_N has

- vertex set $= X$, the input alphabet of N;
- edges $=$ the confusable pairs for N.



Every graph G with vertex set X is of the form $G = G_N$ for some classical channel $N\colon X \rightsquigarrow Y$, for some set Y.

Definition

The *complexity* of G is the smallest possible size of Y.

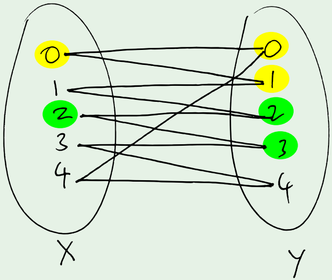
One-shot zero-error capacity $\alpha(N)$

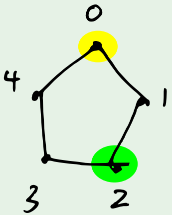
Shannon's one-shot zero-error capacity $\alpha(N)$ is max number of input letters $x_1, \ldots, x_\alpha$ that aren't confusable by N:

- $N(x_i) \cap N(x_j) = \emptyset$ if $i \neq j$; or equivalently,
- $\{x_1, \ldots, x_\alpha\}$ is independent in G_N (i.e., no edge $x_i x_j$ is in G_N)

Example

$\alpha(N_5) = 2$, with independent set $A = \{0, 2\}$.





Shannon zero-error capacity $\Theta(N)$

For $r \geq 1$, write $N^r : X^r \rightsquigarrow Y^r$ for "r parallel uses of N".

Definition

The Shannon zero-error capacity of a classical channel N is

$$\Theta(N) = \lim_{r \to \infty} \alpha(N^r)^{1/r}.$$

Example (Lovász, 1979)

$\Theta(N_5) = \alpha(N_5^2)^{1/2} = \sqrt{5}$.

Θ is generally hard to compute! For example, $\Theta(N_7)$ is unknown.

2. Quantum channels

Quantum channels, non-commutative confusability graphs

Write $M_n = M_n(\mathbb{C})$ and $M_{k\times n} = M_{k\times n}(\mathbb{C})$.

Let $\Phi\colon M_n \to M_k$ be a quantum channel: a completely positive trace-preserving linear map.
Equivalently: Φ is of the form $\Phi(x) = \sum_{i=1}^m A_i x A_i^*$ where $A_1, \dots, A_m \in M_{k\times n}$ with $\sum_{i=1}^m A_i^* A_i = I_n$ are "Kraus operators" for Φ.

Definition (Duan-Severini-Winter 2013)

The non-commutative confusability graph of Φ is

$$\mathcal{S}_\Phi = \operatorname{span}\{A_i^* A_j \colon 1 \le i,j \le m\} \subseteq M_n.$$

$\mathcal{S}_\Phi$ isn't a graph at all! $\mathcal{S}_\Phi$ is an *operator system* in M_n: a unital subspace with $s \in \mathcal{S}_\Phi \iff s^* \in \mathcal{S}_\Phi$.

Example

$A \in M_{k\times n}$ an isometry, $\Phi\colon M_n \to M_k, x \mapsto AxA^* \implies \mathcal{S}_\Phi = \mathbb{C}I_n$.

Why call $\mathcal{S}_\Phi$ a non-commutative graph?

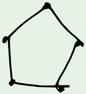
Let G be a graph with vertex set $\{1, 2, \dots, n\}$ and let $E_{i,j}$ be the (i, j) matrix unit in M_n.

Definition

The operator system of G is

$$\mathcal{S}_G = \text{span}\{E_{x,x'} : x, x' \text{ equal or an edge of } G\} \subseteq M_n.$$

Example

$C_5 =$  $\leftrightarrow$ $\mathcal{S}_{C_5} = \left\{ \begin{bmatrix} * & * & 0 & 0 & * \\ * & * & * & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & * & * & * \\ * & 0 & 0 & * & * \end{bmatrix} \right\} \subseteq M_5.$

Identifying G with $\mathcal{S}_G$: graphs become special operator systems.

Non-commutative conf. graphs generalise classical ones

Definition

The quantum version of a classical channel $N\colon X \rightsquigarrow Y$ is

$$\Phi_N\colon M_{|X|} \to M_{|Y|}, \quad \Phi_N(E_{x,x'}) = \delta_{xx'} \frac{1}{|N(x)|} \sum_{y \in N(x)} E_{y,y}.$$

Observation

$\mathcal{S}_{\Phi_N} = \mathcal{S}_{G_N}$

Example

$$\Phi_{N_5}\colon M_5 \to M_5, \quad \begin{bmatrix} z_0 & * & * & * & * \\ * & z_1 & * & * & * \\ * & * & z_2 & * & * \\ * & * & * & z_3 & * \\ * & * & * & * & z_4 \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} z_0+z_4 & 0 & 0 & 0 & 0 \\ 0 & z_0+z_1 & 0 & 0 & 0 \\ 0 & 0 & z_1+z_2 & 0 & 0 \\ 0 & 0 & 0 & z_2+z_3 & 0 \\ 0 & 0 & 0 & 0 & z_3+z_4 \end{bmatrix}$$

has

$$\mathcal{S}_{\Phi_{N_5}} = \left\{ \begin{bmatrix} * & * & 0 & 0 & * \\ * & * & * & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & * & * & * \\ * & 0 & 0 & * & * \end{bmatrix} \right\} = \mathcal{S}_{G_{N_5}}.$$

The quantum complexity of an operator system

Proposition (Duan-Severini-Winter 2013)

Every operator system $\mathcal{S} \subseteq M_n$ is of the form $\mathcal{S} = \mathcal{S}_\Phi$ for some quantum channel $\Phi\colon M_n \to M_k$ and some $k \in \mathbb{N}$.

Definition

The *quantum complexity* of the operator system $\mathcal{S}$ is $\gamma(\mathcal{S}) := \min k$.

Theorem (LPT18 - a connection to spectral graph theory)

For a graph G without isolated vertices, $\gamma(\mathcal{S}_G)$ is the "minimum semidefinite rank" of G: the minimum rank of a positive semidefinite matrix with full support in $\mathcal{S}_G$. (In particular, $\gamma(\mathcal{S}_G) \leq n$.)

Example (for general operator systems, we can have $\gamma(\mathcal{S}) > n$)

$\gamma(\mathcal{C}_2) = 3$ where $\mathcal{C}_2 = \{\left[\begin{smallmatrix} \lambda & a \\ b & \lambda \end{smallmatrix}\right] : \lambda, a, b \in \mathbb{C}\}$.

The quantum subcomplexity of an operator system

Definition

The *quantum subcomplexity* of an operator system $\mathcal{S} \subseteq M_n$ is

$$\beta(\mathcal{S}) = \min\{\gamma(\mathcal{T}) \colon \mathcal{T} \subseteq \mathcal{S} \text{ an operator subsystem}\}.$$

Theorem (L-Paulsen-Todorov 2018)

For a graph G, the number $\beta(\mathcal{S}_G)$ coincides with the orthogonal rank of the complement of G: the smallest k for which there exist $\xi_1, \ldots, \xi_n \in \mathbb{C}^k$ so that

$$\langle \xi_i, \xi_j \rangle \neq 0 \implies ij \text{ is an edge of } G.$$

$\beta(\mathcal{S})$ as a rank minimisation problem

Proposition

If $\mathcal{S} \subseteq M_n$ is an operator system, then $\beta(\mathcal{S}) = \min \operatorname{rank} B$, taken over all $B \in M_m(\mathcal{S})$ with $B \geq 0$ and $\sum_i B_{ii} = I_n$, and all $m \in \mathbb{N}$.

Proposition

The constant-diagonal operator system $\mathcal{C}_n \subseteq M_n$ has $\beta(\mathcal{C}_n) = \lceil \sqrt{n} \rceil$.

Example ($n = 4$)

$$B = \frac{1}{2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 \end{array} \right] \in M_2(\mathcal{C}_4)^+, \text{ so } \beta(\mathcal{C}_4) \leq \operatorname{rank} B = 2.$$

3. Capacity estimates

The quantum zero-error capacity of an operator system

Let $\mathcal{S} \subseteq M_n$ be an operator system.

Definition

The quantum one-shot zero error capacity $\alpha(\mathcal{S})$ is the largest k for which $U\mathcal{S}U^*$ has a $k \times k$ corner containing only diagonal matrices, for some unitary U.

Definition

The quantum zero-error capacity of $\mathcal{S}$ is $\Theta(\mathcal{S}) = \lim_{r\to\infty} \alpha(\mathcal{S}^{\otimes r})^{1/r}$.

- These generalise Shannon: for G the confusability graph of a classical channel N, we have $\alpha(N) = \alpha(\mathcal{S}_G)$ and $\Theta(N) = \Theta(\mathcal{S}_G)$.
- In QIT, it is of interest to compute or estimate $\Theta(\mathcal{S}_\Phi)$ for a quantum channel Φ.

A new upper bound on quantum zero-error capacity

Lemma

β is submultiplicative: $\beta(\mathcal{S}_1 \otimes \mathcal{S}_2) \leq \beta(\mathcal{S}_1)\beta(\mathcal{S}_2)$.

Proof.

$\mathcal{S}_{\Phi_1 \otimes \Phi_2} = \mathcal{S}_{\Phi_1} \otimes \mathcal{S}_{\Phi_2} \implies \gamma$ is submultiplicative; hence β is too. □

Theorem (L-Paulsen-Todorov)

$\alpha(\mathcal{S}) \leq \Theta(\mathcal{S}) \leq \beta(\mathcal{S}) \leq \gamma(\mathcal{S})$ *for any operator system* $\mathcal{S} \subseteq M_n$.

Proof.

First show that $\alpha \leq \beta$; then $\alpha(\mathcal{S}^{\otimes r}) \leq \beta(\mathcal{S}^{\otimes r}) \leq \beta(\mathcal{S})^r$ by Lemma.
Now take $\lim_{r\to\infty}(\cdot)^{1/r}$. □

Open problem: can $\beta(\mathcal{S} \otimes \mathcal{T}) < \beta(\mathcal{S})\beta(\mathcal{T})$ ever occur? (For γ: yes!)

Lovász type bounds on Θ

Definition

The Lovász theta number of a graph G is $\vartheta(G) := \max \|I_n + T\|$, taken over all $T \in \mathcal{S}_G^\perp$ with $I_n + T \geq 0$.

Theorem (Lovász, 1976)

$\Theta(N) \leq \vartheta(G_N)$ *for all classical channels N.*

Definition

The quantum Lovász theta number of an operator system $\mathcal{S} \subseteq M_n$ is $\widetilde{\vartheta}(\mathcal{S}) := \max \|I_{mn} + T\|$ over $m \in \mathbb{N}$, $T \in M_m(\mathcal{S}^\perp)$ with $I_{mn} + T \geq 0$.

Theorem (Duan-Severini-Winter 2013)

$\Theta(\mathcal{S}) \leq \widetilde{\vartheta}(\mathcal{S})$ *for all operator systems* $\mathcal{S}$ *in* M_n.

Moreover, $\widetilde{\vartheta}(\mathcal{S}_G) = \vartheta(G)$*, so this generalises Lovász's theorem.*

β sometimes beats the Lovász $\widetilde{\vartheta}$ bound

Theorem (L-Paulsen-Todorov)

There exist operator systems $\mathcal{R}_t$ for $t \in \mathbb{N}$ so that

$$\Theta(\mathcal{R}_t) \sim \beta(\mathcal{R}_t) \ll \widetilde{\vartheta}(\mathcal{R}_t).$$

Sketch.

Consider

$$\mathcal{R}_t = \mathcal{C}_{t^2} \oplus \mathbb{C}I_t \subseteq M_{t^2+t}$$

where $\mathcal{C}_{t^2} = \{\text{all constant-diagonal matrices in } M_{t^2}\}$. We have

- $\widetilde{\vartheta}(\mathcal{R}_t) \geq \widetilde{\vartheta}(\mathcal{C}_{t^2}) = t^2$,
- $\beta(\mathcal{R}_t) = \beta(\mathcal{C}_{t^2}) + \beta(\mathbb{C}I_t) = t + t = 2t$, and
- $t = \alpha(\mathbb{C}I_t) \leq \Theta(\mathcal{R}_t) \leq \beta(\mathcal{R}_t) = 2t$. □

Thank you!