Tensor algebras of product systems and their C*-envelopes

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Joint work with Adam Dor-On

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History and Motivation

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- A result of Katsoulis and Kribs in that theory is that the C*-envelope of the analytic graph algebra coincides with the Cuntz-Krieger algebra. Davidson and Katsoulis show a similar result for operator algebras associated to "row-contractive" multivariable dynamics.
- In the work of Pimsner, that was later refined by Katsura, it was shown that these constructions share the context of algebras associated to Hilbert bimodules. Katsoulis and Kribs proved that in this general context, Katsura's Cuntz-Pimsner algebra is always the C*-envelope of an analytic tensor algebra of the Hilbert bimodule.

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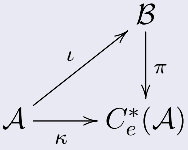
Definition (Arveson; exists by work of Hamana)

The C-envelope of an operator algebra $\mathcal{A}$ is a C*-cover $(C_e^*(\mathcal{A}), \kappa)$ such that for any other C*-cover $(\mathcal{B}, \iota)$, there is a surjective $*$-homomorphism $\pi : \mathcal{B} \to C_e^*(\mathcal{A})$ such that the following diagram*



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$$\begin{array}{ccc} & & \mathcal{B} \\ & \overset{\iota}{\nearrow} & \downarrow \pi \\ \mathcal{A} & \xrightarrow[\kappa]{} & C_e^*(\mathcal{A}) \end{array}$$

commutes.

Caution: C*-envelope$\neq$ enveloping C*-algebra

A simple-minded approach to C*-correspondences

A (represented) C*-correspondence consists of a pair $(X, \mathcal{C})$ with $\mathcal{C} \subseteq B(\mathcal{H})$ a C*-algebra acting non-degenerately and $X \subseteq B(\mathcal{H})$ a closed $\mathcal{C}$-bimodule satisfying

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(ii) $\mathcal{H} = \ell^2(\mathbb{N})$, $\mathcal{C} = \mathbb{C}I$, $X = \mathbb{C}V$, where V is the forward shift.

A simple-minded approach to C*-correspondences

(iii) $\mathcal{H}$ any Hilbert space, $V_1, V_2, \ldots, V_n$ any collection of isometries with orthogonal ranges. Here $\mathcal{C} = \mathbb{C}I$ and $X = [\{V_1, V_2, \ldots V_n\}]$

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(iv) Let $\mathcal{C}$ be a unital C*-algebra and α any $*$-automorphism of $\mathcal{C}$. Consider the (represented) crossed product C*-algebra $\mathcal{C} \rtimes_\alpha \mathbb{Z}$ generated by $\mathcal{C}$ and a unitary U implementing α, i.e., $cU = U\alpha(c)$.

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Definition of a Hilbert C*-module

Let $\mathcal{C}$ be a C*-algebra. An *inner-product right $\mathcal{C}$-module* is a linear space X which is a right $\mathcal{C}$-module together with a map

$$(\cdot,\cdot) \mapsto \langle \cdot,\cdot \rangle : X \times X \to \mathcal{C}$$

such that

$$\begin{aligned} \langle x, \lambda y + z \rangle &= \lambda \langle x, y \rangle + \langle x, z \rangle \\ \langle x, yc \rangle &= \langle x, y \rangle c \\ \langle y, x \rangle &= \langle x, y \rangle^* \\ \langle x, x \rangle &\geq 0; \text{ if } \langle x, x \rangle = 0 \text{ then } x = 0. \end{aligned}$$

For $x \in X$ we write $\|x\|_X^2 := \|\langle x, x \rangle\|_{\mathcal{C}}$. If X equipped with that norm is a complete normed space then it will be called *Hilbert $\mathcal{C}$-module*.

Definition of a C*-correspondence

For a Hilbert $\mathcal{C}$-module X we define the set $L(X)$ of the *adjointable maps* that consists of all maps $s : X \to X$ for which there is a map $s^* : X \to X$ such that

$$\langle sx, y \rangle = \langle x, s^* y \rangle$$

for all $x, y \in X$.

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If φ is injective then the C*-correspondence $(X, \mathcal{C}, \varphi)$ is said to be injective.

Representations of a C*-correspondence

A (Toeplitz) representation (π, ψ) of X into a C*-algebra $\mathcal{B}$, is a pair of a $*$-homomorphism $\pi\colon \mathcal{C} \to B$ and a linear map $\psi\colon X \to \mathcal{B}$, such that

(i) $\pi(c)\psi(x) = \psi(\varphi_X(c)(x)) = \psi(cx)$,

(ii) $\psi(x)^*\psi(y) = \pi(\langle x, y\rangle)$,

for $c \in \mathcal{C}$ and $x, y \in X$. An easy application of the C*-identity shows that

(iii) $\psi(x)\pi(c) = \psi(xc)$

is also valid.

A representation (π, ψ) is said to be *injective* iff π is injective; in that case ψ is an isometry.

The Toeplitz algebras

Definition

The Toeplitz-Cuntz-Pimsner C*-*algebra* $\mathcal{T}_X$ *of a* C*-*correspondence* $(X, \mathcal{C})$ *is the* C*-*algebra generated by all elements of the form* $\pi_\infty(c), \psi_\infty(x)$, $c \in \mathcal{C}$, $x \in X$, *where* $(\pi_\infty, \psi_\infty)$ *denotes the universal Toeplitz representation of* $(X, \mathcal{C})$.

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Definition

The tensor algebra $\mathcal{T}_X^+$ *is the norm-closed subalgebra of* $\mathcal{T}_X$ *generated by all elements of the form* $\pi_\infty(c), \psi_\infty(x)$, $c \in \mathcal{C}$, $x \in X$, *where* $(\pi_\infty, \psi_\infty)$ *denotes the universal Toeplitz representation of* $(X, \mathcal{C})$.

An (incomplete) example: $\mathcal{C}_\alpha$

Let $\mathcal{C}$ be a unital C*-algebra and let α be an $*$-automorphism of $\mathcal{C}$. The C*-algebra $\mathcal{C}$ (right) acts on $\mathcal{C}_\alpha := \mathcal{C}$ by multiplication. Then $\mathcal{C}_\alpha$ equipped with the inner product

$$\langle x, y \rangle := x^* y, \quad x, y \in \mathcal{C}_\alpha$$

becomes a Hilbert $\mathcal{C}$-module. The left action of $\mathcal{C}$ on X_α is given by $\varphi(c)x := \alpha(c)x$, $x \in X_\alpha$, $a \in \mathcal{C}$.
Given any representation (π, ψ) we have

$$\pi(c)\psi(1) = \psi(c \cdot 1) = \psi(\alpha(c)) = \psi(1)\pi(\alpha(c))$$

and

$$\psi(1)^*\psi(1) = \pi(\langle 1, 1 \rangle) = \pi(1) = I.$$

Hence $\mathcal{T}_{\mathcal{C}_\alpha}$ is the extension by the compacts of the crossed product C*-algebra $\mathcal{C} \rtimes_\alpha \mathbb{Z}$.

The Cuntz-Pimsner algebra of a C*-correspondence

The compact operators $K(X) \subseteq L(X)$ is the closed subalgebra of $L(X)$ generated by the rank one operators

$$\theta_{x,y}(z) := x \langle y, z \rangle \,, \quad x, y, z \in X.$$

Let (π, ψ) be a representation of $(X, \mathcal{C}, \varphi)$. Then there exists a map

$$\psi^{(1)} \colon K(X) \longrightarrow \mathrm{C}^*(\pi, \psi)$$

so that $\psi_t(\theta_{x,y}) = \psi(x))\psi(y)^*$, for all $x, y \in X$.

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A representation (π, ψ) *of a* C*-*correspondence* $(X, \mathcal{C}, \varphi)$ *is said to be a covariant representation iff*

$$\pi(c) = \psi^{(1)}(\varphi(c)), \quad \text{for all } c \in J_X,$$

where $J_X = \varphi^{-1}(K(X)) \cap (\ker \varphi)^{\perp}$.

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$\mathcal{C}_\alpha$ revisited

Now $\varphi(\mathcal{C}) = K(\mathcal{C}_\alpha)$ and $\ker\varphi = \{0\}$. Hence $J_{\mathcal{C}_\alpha} = \mathcal{C}$.
Looking at the unit $1 \in \mathcal{C}$, we see that $\varphi(1) = \theta_{1,1}$ and so for a covariant representation (π, ψ) of $\mathcal{C}_\alpha$ we have

$$I = \pi(1) = \psi^{(1)}(\theta_{1,1}) = \psi(1)\psi(1)^*.$$

This is also sufficient for covariance. Hence

$$\mathcal{O}_{\mathcal{C}_\alpha} \simeq \mathcal{C} \rtimes_\alpha \mathbb{Z}$$

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Theorem (Fowler 02, Katsura 04)

If $(X, \mathcal{C})$ is a (represented) C-correspondence then*

$$\mathcal{T}_X \simeq \mathrm{C}^*(\mathcal{C} \otimes I, X \otimes V) \subseteq B(\mathcal{H} \otimes \ell^2(\mathbb{N})),$$

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What about $\mathcal{O}_X$?

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Theorem (Katsoulis & Kribs 2006)

If $(X, \mathcal{C})$ *is a* C*-*correspondence then*

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Significant applications of this approach came only recently with the resolution of the Hao-Ng isomorphism for the reduced crossed product. (Katsoulis 2016)

Hao-Ng isomorphism

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Let $(X, \mathcal{C})$ be a C*-correspondence and $\mathcal{G}$ a locally compact Hausdorff group. An action $\alpha : \mathcal{G} \to \operatorname{Aut} \mathcal{O}_X$ such that $\alpha_g(X) = X$ and $\alpha_g(\mathcal{C}) = \mathcal{C}$ is called a *generalized gauge action*.

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For instance in the case where $\mathcal{G}$ is discrete...

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(iii) Bedos, Kaliszewski, Quigg, Robertson (2015): $(X, \mathcal{C})$ is arbitrary, $\mathcal{G}$ is discrete and exact. (Katsura?)

Hao-Ng isomorphism

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Based on joint work with Chris Ramsey

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Theorem (K., 2017)

Let $(X, \mathcal{C})$ be a C*-*correspondence and* $\mathcal{G}$ *a discrete group. If* $\alpha : \mathcal{G} \to \operatorname{Aut} \mathcal{O}_X$ *is a generalized gauge action, the Hao-Ng isomorphism holds, i.e.,*

$$\mathcal{O}_X \rtimes_\alpha^r \mathcal{G} \simeq \mathcal{O}_{X \rtimes_\alpha^r \mathcal{G}}$$

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Based on joint work with Chris Ramsey

Theorem (K., 2017)

Let $(X, \mathcal{C})$ *be a* C*-*correspondence and* $\mathcal{G}$ *a discrete group. If* $\alpha : \mathcal{G} \to \operatorname{Aut} \mathcal{O}_X$ *is a generalized gauge action, the Hao-Ng isomorphism holds, i.e.,*

$$\mathcal{O}_X \rtimes_\alpha^r \mathcal{G} \simeq \mathcal{O}_{X \rtimes_\alpha^r \mathcal{G}}$$

In forthcoming work with Chris Ramsey we address arbitrary groups and obtain a very general result that subsumes that of Kim.

Hao-Ng isomorphism

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Ingredients of the proof

Theorem (K. 2016)

If $(\mathcal{A}, \mathcal{G}, \alpha)$ *is a discrete dynamical system with* $\mathcal{A}$ *an arbitrary approximately unital operator algebra, then*

$$\mathrm{C}^*_e\big(\mathcal{A} \rtimes^r_\alpha \mathcal{G}\big) \simeq \mathrm{C}^*_e(\mathcal{A}) \rtimes^r_\alpha \mathcal{G}.$$

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Theorem (K., Ramsey, Memoirs AMS 201?)

Let $(X, \mathcal{C})$ *be a* C^**-correspondence and* $\mathcal{G}$ *any locally compact group. If* $\alpha : \mathcal{G} \to \operatorname{Aut} \mathcal{O}_X$ *is a generalized gauge action, then*

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Then the proof ...

Product systems: a simple-minded approach

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Definition

A product system $X = \{X_p\}_{p \in P}$ over P with coefficients in a C**-algebra $\mathcal{C}$ is a collection of* C**-correspondences represented on the same Hilbert space such that*

- $X_e = \mathcal{C}$,
- $X_p X_q \subseteq X_{pq}$ *total, for all* $p, q \in P$.

Remarks

(i) Any C*-correspondence $(X, \mathcal{C})$ generates a product system $X = \{X_n\}_{n \in \mathbb{N}}$ by taking

$$X_n := \overline{[X^n]}, \quad n \in \mathbb{N}.$$

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Furthermore, all product systems over $\mathbb{N}$ arise this way.
(In the pertinent literature $\overline{[XY]}$ is nothing but the internal tensor product $X \otimes Y$.)
(ii) The *trivial product system* $X = \{X_p\}_{p \in P}$ *over* P is the one satisfying $X_p = \mathbb{C}$, for all $p \in P$.
(iii) We would like a representation theory for product systems that allows for an identification of the form

$$\mathcal{T}_X \simeq \mathrm{C}^*(\mathcal{C} \otimes I, X \otimes V) \subseteq B(\mathcal{H} \otimes \ell^2(P)),$$

where $V : P \to B(\ell^2(P))$ is the left regular representation of P.

Product systems over abelian ordered groups

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An abelian lattice-ordered group is a pair (G, P) with G abelian, generated by a unital semigroup P such that $P \cap P^{-1} = \{0\}$, so that with the induced partial order there are least upper bounds $s \vee t$ and greatest lower bounds $s \wedge t$ for any $s, t \in G$.

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2. For $p, q \in P$ there are $\mathcal{A}$-linear unitary products $M_{p,q} : X_p \otimes X_q \to X_{pq}$ that are associative in the sense that $M_{p,qr}(I_{X_p} \otimes M_{q,r}) = M_{pq,r}(M_{p,q} \otimes I_{X_r})$.

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Example (Semigroup dynamical systems)

Suppose $\alpha : P \to \mathrm{End}(\mathcal{C})$ *is non-degenerate. By taking* $X_p = \mathcal{C}$ *and setting* $M_{p,q}(a \otimes b) = \alpha_q(a)b$ *we get a product system.*

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Let $X = (X_p)$ be a product system over an abelian lattice-ordered semigroup P, with coefficients in $\mathcal{C}$. We say that X is compactly aligned if whenever $p, q \in P$ and K_p, K_q are compact operators on X_p and X_q, respectively, then $(K_p \otimes I_{p^{-1}(p \vee q)})(K_q \otimes I_{q^{-1}(p \vee q)})$ is a compact operator on $X_{p \vee q}$.

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If $\mathcal{C}$ acts on the left on all X_p by compact operators then X is compactly aligned

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$$\psi^{(p)}(K_p)\psi^{(q)}(K_q) = \psi^{(p\vee q)}((K_p \otimes I_{p^{-1}(p\vee q)})(K_q \otimes I_{q^{-1}(p\vee q)}))$$

for $K_p \in \mathcal{K}(X_p)$ and $K_q \in \mathcal{K}(X_q)$.

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When X is a *$\tilde{\varphi}$-injective* product system over a lattice-ordered abelian group (G, P), there is also a Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ introduced by Sims and Yeend.

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The definition of CNP representations is in the spirit of Katsura’s analogous definition for covariant representations of C*-correspondences but far more complicated....

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Lemma

Suppose X is a compactly aligned product system over an abelian lattice-ordered group (G, P). Let ψ be a CNP representation of X and let $p \mapsto U_p$ be a unitary representation of P. Then $\psi \otimes U$ is a CNP representation.

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Carlsen, Larsen, Sims and Vittadello have shown that the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ is co-universal w.r.t. Nica-covariant, *gauge-compatible* isometric representations of X that are faithful on $\mathcal{A}$.

The tensor algebra

One can then ask for the C*-envelope of the non-self-adjoint analytic subalgebra

$$\mathcal{NT}_X^+ = \overline{\mathrm{Alg}}\{\iota_p(x)\}_{p \in P, x \in X_p}.$$

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Theorem (Dor-On & Katsoulis)

Suppose X is $\tilde{\varphi}$-injective, compactly aligned product system over an abelian lattice-ordered group (G,P). Then $C_e^*(\mathcal{NT}_X^+)$ is $\mathcal{NO}_X$.

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3. Since $\lambda : P \to B(\ell^2(P))$ dilates to a unitary representation $\hat{\lambda} : P \to B(\ell^2(G))$, the representation $\psi \otimes \hat{\lambda}$ dilates $\psi \otimes \lambda$, and promotes to a *representation of $\mathcal{NO}_X$*. Hence, since $(\psi \otimes \hat{\lambda})_*|_{\mathcal{NT}_X^+}$ dilates $(\psi \otimes \lambda)_*|_{\mathcal{NT}_X^+}$, we see that the range of the representation $(\psi \otimes \hat{\lambda})_{**} : \mathcal{NO}_X \to B(\mathcal{H} \otimes \ell^2(G))$ contains a completely isometric copy of $\mathcal{NT}_X^+$. *(!)*

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4. The gauge-invariant uniqueness theorem of Carlsen, Larsen, Sims and Vitadello then finishes the proof.

Higher rank graphs

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Let $G = (V, E, r, s)$ be a countable directed graph. Divide $E = E_1 \cup ... \cup E_d$ into d colors, so that every path $\lambda \in E^\bullet$ has a multidegree $d(\lambda)$.

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A subset $B \subseteq v\Lambda$ is called *exhaustive* if for every $\lambda \in v\Lambda$ there's $\mu \in B$ such that $\Lambda^{min}(\lambda, \mu) \neq \emptyset$. *(Sketch some drawings)*

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We say S is *Cuntz-Krieger* if additionally

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Denote by $\mathcal{O}(\Lambda)$ the universal C*-algebras generated by a CK Λ-family.

Higher rank CK families

A set of partial isometries $S := \{S_\lambda\}_{\lambda \in \Lambda}$ for a higher rank graph (Λ, d) is called a *Toeplitz-Cuntz-Krieger* Λ-family if

(P) $(S_v)_{v \in V}$ are pairwise orthogonal projections,

(C) $S_\lambda S_\mu = \delta_{s(\lambda),r(\mu)} S_{\lambda\mu}$ for $\lambda, \mu \in \Lambda$,

(DC) $S_\lambda^* S_\mu = \sum_{(\alpha,\beta) \in \Lambda^{min}(\lambda,\mu)} S_\alpha S_\beta^*$.

We say S is *Cuntz-Krieger* if additionally

(CK) $\prod_{\lambda \in B}(S_v - S_\lambda S_\lambda^*) = 0$ for all $v \in V$ and $\emptyset \neq B \subseteq v\Lambda$ finite exhaustive set.

Denote by $\mathcal{O}(\Lambda)$ the universal C*-algebras generated by a CK Λ-family. A theorem of Katsoulis and Kribs shows that when Λ is *row-finite and sourceless*, we have that $\mathcal{O}(\Lambda)$ is the C*-envelope of the analytic graph tensor algebra $\mathcal{T}^+(\Lambda)$.

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Theorem (Davidson, Fuller & Kakariadis)

Let $\alpha : \mathbb{Z}^d_+ \to \mathrm{End}(A)$ be a non-degenerate C-dynamical system. Then $C^*_e(\mathcal{N}\mathcal{T}^+(A,\alpha)) \cong \mathcal{N}\mathcal{O}(A,\alpha)$ as before.*

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Here there is no $\tilde{\varphi}$-injectivity, as this is automatic for $(\mathbb{Z}^d, \mathbb{N}^d)$.

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Let X be a product system over an abelian lattice ordered group (G, P). An action $\alpha : \mathcal{G} \to \mathrm{Aut}\, \mathcal{NT}_X$ such that $\alpha_g(X_p) = X_p$ for all $g \in \mathcal{G}$ and $p \in P$ is called a *generalized gauge action*.

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Theorem (Dor-On & Katsoulis)

Suppose X is a finitely aligned $\tilde{\varphi}$-injective product system over an abelian lattice-ordered (G, P), and $\alpha : \mathcal{G} \to \operatorname{Aut}(\mathcal{NT}_X)$ a gen. discrete group gauge action. Then $\mathcal{NO}_X \rtimes_\alpha \mathcal{G} \cong \mathcal{NO}_{X \rtimes_\alpha \mathcal{G}}$.

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Thus, $C^*_e(\mathcal{NT}_X^+)$ satisfies a gauge-invariant uniqueness theorem.

Ending

Thank you for your attention !