Interpolating sequences and Kadison–Singer

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Definition

A sequence (z_n) in $\mathbb{D}$ is interpolating for H^∞ if for every sequence $(\lambda_n) \in \ell^\infty$, there exists $f \in H^\infty$ with

$$f(z_n) = \lambda_n \quad (n \in \mathbb{N}).$$

Write (z_n) satisfies (IS).

Why study interpolating sequences?

Let

$$\mathfrak{M} = \{\rho : H^\infty \to \mathbb{C} : \rho \text{ is linear, multiplicative}\} \setminus \{0\}$$

and identify $\mathbb{D} \subset \mathfrak{M}$ via point evaluations.

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Proposition

If (z_n) is an interpolating sequence, then $\overline{\{z_n : n \in \mathbb{N}\}} \subset \mathfrak{M}$ is homeomorphic to $\beta\mathbb{N}$. In particular, $\mathfrak{M}$ is not metrizable and has cardinality $2^{2^{\aleph_0}}$.

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An analytic disc in $\mathfrak{M}$ is the image of a continuous injection $L : \mathbb{D} \to \mathfrak{M}$ such that $\widehat{f} \circ L$ is analytic for every $f \in H^\infty$.

Theorem (Hoffman, 1967)

A point $m \in \mathfrak{M}$ lies in an analytic disc if and only if it belongs to the closure of an interpolating sequence.

Carleson's interpolation theorem

A sequence (z_n) in $\mathbb{D}$

(SS) is strongly separated if there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $f_k \in H^\infty$ with $||f_k||_\infty \leq 1$ and $f_k(z_j) = \varepsilon \delta_{kj}$.

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(WS) is weakly separated if there exists $\varepsilon > 0$ such that whenever $j \neq k$, there exists $f_{kj} \in H^\infty$ with $||f_{kj}||_\infty \leq 1$ and $f_{kj}(z_j) = 0$ and $f_{kj}(z_k) = \varepsilon$. Equivalently, (z_n) is separated in the Poincaré metric of $\mathbb{D}$.

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(C) satisfies the Carleson measure condition if there exists $M > 0$ so that

$$\sum_j (1 - |z_j|^2)|f(z_j)|^2 \le M \int_{\partial \mathbb{D}} |f|^2 \, dm \quad \text{for all } f \in \mathbb{C}[z].$$

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Example

If $|z_n| \to 1$ exponentially fast, then (z_n) is an interpolating sequence.

A Hilbert space proof

Shapiro–Shields (1962): Different proof of Carleson's theorem, reformulate (IS) in terms of the Hardy space

$$H^2 = \Big\{ f = \sum_{n=0}^\infty a_n z^n \in \mathcal{O}(\mathbb{D}) : \sum_{n=0}^\infty |a_n|^2 < \infty \Big\}.$$

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H^2 is a Hilbert function space on $\mathbb{D}$: For all $f \in H^2$ and $w \in \mathbb{D}$,

$$f(w) = \langle f, K(\cdot, w) \rangle_{H^2}, \quad \text{where} \quad K(z,w) = \frac{1}{1 - z\overline{w}}$$

is the reproducing kernel of H^2.

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is the reproducing kernel of H^2. The multiplier algebra is

$$\operatorname{Mult}(H^2) = \{ \varphi : \mathbb{D} \to \mathbb{C} : \varphi \cdot f \in H^2 \text{ for all } f \in H^2 \},$$

normed by $||\varphi||_{\operatorname{Mult}(H^2)} = ||f \mapsto \varphi \cdot f||_{B(H^2)}$.

Fact

$\operatorname{Mult}(H^2) = H^\infty$ with equality of norms.

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Key property

H^2 and $\mathcal{D}$ are complete Pick spaces.

Complete Pick spaces

Let $\mathcal{H}$ be a Hilbert function space on X with kernel K. $\mathcal{H}$ is a complete Pick space if

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for some function $b : X \to \ell_2$.

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Examples of complete Pick spaces

- Hardy space H^2, where $K(z, w) = \frac{1}{1 - z\overline{w}}$,
- Dirichlet space $\mathcal{D}$, where $K(z, w) = -\frac{1}{z\overline{w}} \log(1 - z\overline{w})$ (Agler, 1988).
- Drury-Arveson space H^2_d on the unit ball $\mathbb{B}_d \subset \mathbb{C}^d$ (a.k.a. symmetric Fock space) with kernel

$$K(z, w) = \frac{1}{1 - \langle z, w \rangle}.$$

The Bergman space $L^2_a = \mathcal{O}(\mathbb{D}) \cap L^2(\mathbb{D})$ is not a complete Pick space.

Interpolating sequences for complete Pick spaces

Let $\mathcal{H}$ be a complete Pick space on X with kernel K. A sequence (z_n) in X

(IS) is an interpolating sequence if for every sequence $(\lambda_n) \in \ell^\infty$, there exists $\varphi \in \text{Mult}(\mathcal{H})$ with

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Known results about interpolating sequences

Easy facts

In general, (IS) $\Rightarrow$ (SS) and (IS) $\Rightarrow$ (WS) + (C).

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Theorem (Bøe, 2005)

(WS) + (C) $\Leftrightarrow$ (IS) for every space on the unit ball $\mathbb{B}_d$ with kernel

$$K(z,w) = \frac{1}{(1 - \langle z, w \rangle)^\alpha}, \quad \text{where } \alpha \in (0,1).$$

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Theorem (Agler–McCarthy, 2002)

(WS) + (C) $\Rightarrow$ (SS) for every complete Pick space.

The main result

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In every complete Pick space, (WS) + (C) $\Leftrightarrow$ (IS).

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In every complete Pick space, (WS) + (C) $\Leftrightarrow$ (IS).

In this case, there exists a linear operator of operator of interpolation, i.e.

$$\mathrm{Mult}(\mathcal{H}) \to \ell^\infty, \quad \varphi \mapsto (\varphi(z_n)),$$

has a bounded linear right-inverse.

Grammians

Let $\mathcal{H}$ be a complete Pick space on X with kernel K, let (z_n) be a sequence in X. Let $k_i = K(\cdot, z_i)$ and let

$$G[(z_n)] = \Big[\Big\langle \frac{k_i}{||k_i||}, \frac{k_j}{||k_j||} \Big\rangle \Big]_{i,j}$$

be the Grammian.

Proposition

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Theorem (Marshall–Sundberg, 1994)

(z_n) satisfies (IS) iff $G[(z_n)]$ is bounded and bounded below.

Kadison–Singer

Question (Kadison–Singer, 1959)

Let $D \subset B(\ell^2)$ be the algebra of all diagonal operators. Does every pure state on D admit a unique extension to a state on $B(\ell^2)$?

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Equivalent statement (Feichtinger conjecture):

Theorem (Marcus–Spielman–Srivastava, 2013)

Let (v_n) be a sequence of unit vectors in a Hilbert space and let $G = [\langle v_i, v_j \rangle]_{i,j}$ be the Grammian. If G is bounded, then (v_n) is a finite union of sequences whose Grammian is bounded and bounded below.

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Corollary

If (z_n) satisfies (C), then (z_n) is a finite union of sequences that satisfy (IS).

Idea of the proof of (WS) + (C) $\Rightarrow$ (IS)

Assume (x_n) and (y_n) satisfy (IS), their union satisfies (WS) + (C).

Goal

If $(\lambda_n), (\mu_n) \in \ell^\infty$, find $\varphi \in \text{Mult}(\mathcal{H})$ with $\varphi(x_n) = \lambda_n$ and $\varphi(y_n) = \mu_n$.

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Theorem 1 (Agler–McCarthy, 2002)

$G[(x_n)]$ is bounded below iff there exists a sequence (φ_n) in $\mathrm{Mult}(\mathcal{H})$ such that $[M_{\varphi_1} \; M_{\varphi_2} \; \cdots]$ is bounded and such that $\varphi_k(x_n) = \delta_{nk}$.

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(z_n) satisfies (WS) + (C) iff there exists a sequence (φ_n) in $\mathrm{Mult}(\mathcal{H})$ such that $[M_{\varphi_1}\ M_{\varphi_2}\ \cdots]^T$ is bounded and such that $\varphi_k(z_n) = \delta_{nk}$.

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where $\varphi_n(x_n) = 1 = \psi_n(y_n)$ and $\theta_k(x_n) = \delta_{nk}$ and $\omega_k(x_n) = 0$.

Then $\varphi \in \text{Mult}(\mathcal{H})$ with

$$\varphi(x_n) = \lambda_n \quad \text{and} \quad \varphi(y_n) = \mu_n.$$

Column multipliers and row multipliers

The proof does not require the Marcus–Spielman–Srivastava theorem for every space $\mathcal{H}$ with the following property:

Property (BC) ⇒ (BR)

For all sequences (φ_n) in $\mathrm{Mult}(\mathcal{H})$,

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Question

Does every complete Pick space satisfy (BC) $\Rightarrow$ (BR)?

Column multipliers and row multipliers

The proof does not require the Marcus–Spielman–Srivastava theorem for every space $\mathcal{H}$ with the following property:

Property (BC) $\Rightarrow$ (BR)

For all sequences (φ_n) in Mult$(\mathcal{H})$,

$$\begin{bmatrix} M_{\varphi_1} \\ M_{\varphi_2} \\ \vdots \end{bmatrix} \text{ bounded } \quad \Rightarrow \quad [M_{\varphi_1} \, M_{\varphi_2} \cdots] \text{ bounded.}$$

This property is satisfied by

- the Dirichlet space (Trent),
- H^2_d for $d < \infty$ (Aleman–H.–McCarthy–Richter).

Question

Does every complete Pick space satisfy (BC) $\Rightarrow$ (BR)? An application

Thank you!

Non-essentially normal multipliers

An operator $T \in \mathcal{B}(\mathcal{H})$ is essentially normal if $TT^* - T^*T$ is compact.

Easy fact

There exists a multiplication operator M_φ on H^2 which is an isometry with infinite dimensional cokernel. In particular, M_φ is not essentially normal.

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Proposition (Aleman, H., McCarthy, Richter)

Let $\mathcal{H}$ be a complete Pick space on a connected topological space X with jointly continuous kernel K. If K is unbounded, then there exists a multiplication operator which is not essentially normal.

This applies to every complete Pick space mentioned in this talk.

Thank you!