Model theory of correspondences

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Preamble

- With 20-20 hindsight, model theoretic functional analysis began with Krivine and the model theory of Banach spaces in the 1960's.
- The logical understanding of this was provided by Henson and positive bounded logic in the mid-70's.
- Long pause until the introduction of continuous model theory first by Ben Ya'acov and then together with Henson in about 2005.
- This led to the first wave of results in the model theory of operator algebras: C^*-algebras, tracial von Neumann algebras, II_1 factors all form elementary classes; the resolution of McDuff's conjecture; the recognition of nuclearity as a forcing condition; various results about the language of these classes (mostly negative)

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- What is the second wave? Look for problems where model theoretic techniques can be applied. How do you do that?
- Slogan: Ultraproducts are model theory!

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- What is the second wave? Look for problems where model theoretic techniques can be applied. How do you do that?
- Slogan: Ultraproducts are model theory!
- Secret agenda: Explain how we capture the notion of correspondence.
- Explain why we want to do this.
- Convince you this is a reasonable thing to do.

Outline of talk

- Basics of correspondences
- Basics of continuous model theory
- The role of ultraproducts
- Model theoretic case study: Property T for II_1 factors
- The main theorem

Crash course on correspondences

Fix tracial von Neumann algebras (M, τ_M) and (N, τ_N).

- An M-N correspondence is a Hilbert space H together with commuting normal representations π_M and π_N.
- If $\phi : M \to N$ is a completely positive map then on $M \otimes N$ define an inner product via

$$\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle = \tau_N(\phi(a_2^* a_1) b_2^* b_1)$$

  and let H_ϕ be the correspondence obtained by taking the completion of $M \otimes N$ with respect to $\langle \cdot, \cdot \rangle$.
- If H is a correspondence, $\xi \in H$ is K-bounded if for all $c \in M$, $d \in N$

$$\langle c\xi, c\xi \rangle \leq K\tau_M(c) \text{ and } \langle \xi d, \xi d \rangle \leq K\tau_N(d).$$

Crash course, cont'd

- Suppose H is a correspondence and $\xi \in H$ is left bounded and $R_\xi : N \to H$ by the right action. We define $\phi_\xi : M \to N$ by
$$\phi_\xi(m) = R_\xi^* m R_\xi .$$
ϕ_ξ is a c.p. map.
- If ξ is a left bounded vector in a correspondence H then H_{ϕ_ξ} is isomorphic to $\overline{M\xi N}$ via the map which sends $1 \otimes 1$ to ξ.
- Every correspondence is the direct sum of cyclic correspondences of the form H_ϕ where ϕ is a c.p. map associated to a 1- bounded vector.

One slide on continuous model theory

A language $\mathcal{L}$ has sorts, functions and relations. An $\mathcal{L}$-structure is an interpretation of the language subject to the following:

- Sorts are complete, bounded metric spaces with designated metrics.
- Functions are defined on sorts and are uniformly continuous; the language knows the uniform continuity modulus.
- Relations are defined on sorts, bounded and real-valued. They are also uniformly continuous and this is known to the language.
- Terms are formed by composing functions; formulas are defined inductively by:
  - $R(\tau_1, \ldots, \tau_n)$ where R is a relation and $\tau_1, \ldots, \tau_n$ are terms,
  - if $f \colon R^n \to R$ is continuous and $\varphi_1, \ldots, \varphi_n$ are formulas then $f(\varphi_1, \ldots, \varphi_n)$ is a formula, and
  - $\sup_x \varphi$ and $\inf_x \varphi$ are formulas if φ is.

Examples

- In the theory of tracial von Neumann algebras, $\max\{\|x - x^*\|_2, \|x^2 - x\|_2\}$ is a formula. It evaluates to 0 iff x is a projection.
- Consider, for $k, n \in \mathbb{N}$, the function

$$R_n^k(\bar{x}) = \inf_{\varphi,\psi} \|\bar{x} - \psi(\varphi(\bar{x}))\|$$

  where $\varphi\colon A \to M_n(\mathbb{C})$ and $\psi\colon M_n(\mathbb{C}) \to A$ range over cpc maps. Very non-trivially this can be expressed as a formula in the language of C*-algebras.
- A sentence is a formula with no free variables. The theory of an $\mathcal{L}$-structure is the set of sentences which evaluate to 0 in that structure.

Ultraproducts

- Suppose $\mathcal{U}$ is an ultrafilter on a set I and $\overline{r} = \langle r_i : i \in I \rangle$ is an I-indexed family of real numbers. We define the ultralimit of $\overline{r}$ with respect to $\mathcal{U}$ as follows:

$$\lim_{i \to \mathcal{U}} r_i = r \text{ iff for every } \epsilon > 0, \{i \in I : |r - r_i| < \epsilon\} \in \mathcal{U}.$$

- If (X_i, d_i) is an I-indexed sequence of uniformly bounded metric spaces. Define the pseudo-metric d on $\prod_{i \in I} X_i$ as follows:

$$d(\overline{x}, \overline{y}) = \lim_{i \to \mathcal{U}} d_i(x_i, y_i).$$

- The metric ultraproduct of the X_i's with respect to $\mathcal{U}$, $\prod_{\mathcal{U}} X_i$, is the metric space obtained by quotienting $\prod_{i \in I} X_i$ by d. If all the X_i's are equal to a fixed X we will often write $X^{\mathcal{U}}$ for this ultraproduct and call it the ultrapower.

Ultraproducts, cont'd

Suppose that $\mathcal{L}$ is a language and M_i is an $\mathcal{L}$-structure for each $i \in I$; fix $\mathcal{U}$ an ultrafilter on I. We create $\prod_{\mathcal{U}} M_i$ as follows:

- If S is a sort, form $\prod_{\mathcal{U}} S(M_i)$.
- Define functions coordinatewise on sorts.
- For a relation R and $\bar{a}$ from an appropriate sort, let

$$R(\bar{a}) = \lim_{i \to \mathcal{U}} R^{M_i}(\bar{a}_i).$$

The role of ultraproducts

Theorem (Łoś' Theorem)

Suppose $\mathcal{M}_i$ are $\mathcal{L}$-structures for all $i \in I$, $\mathcal{U}$ is an ultrafilter on I, $\varphi(\overline{x})$ is an $\mathcal{L}$-formula and $\overline{a} \in \mathcal{M} := \prod_{\mathcal{U}} \mathcal{M}_i$ then

$$\varphi^{\mathcal{M}}(\overline{a}) = \lim_{i \to U} \varphi^{\mathcal{M}_i}(\overline{a}_i).$$

Theorem

For a class of $\mathcal{L}$-structures $\mathcal{C}$, TFAE

- *$\mathcal{C}$ is an elementary class i.e. all $\mathcal{L}$-structures satisfying some set of sentences.*
- *$\mathcal{C}$ is closed under isomorphisms, ultraproducts and ultraroots.*

Definable sets via functors

- Met will be the category of bounded metric spaces with isometries as morphisms. Mod(T) is the category of models of T.
- Suppose we have a theory T in a language $\mathcal{L}$ and S_i for $i \leq n$ are sorts in $\mathcal{L}$. We call a functor

$$X\colon \mathrm{Mod}(T) \to \mathrm{Met}$$

  a T-functor if for every model $\mathcal{M}$ of T, $X(\mathcal{M})$ is a closed subset of $\prod_{j=1}^{m} S_j(\mathcal{M})$ and X is just restriction on morphisms.
- This functor is called a *definable set* if for all models $\mathcal{M}$ of T the function $d(x, X(\mathcal{M}))$ is a formula (uniformly definable predicate) in T.

Examples

- The set of projections in a C^*-algebra.
- The set of self-adjoint contractions but not the set of normal contractions in a C^*-algebra.
- The set of pairs of projections that are Murray-von Neumann equivalent in a II_1 factor.

Definable sets and ultraproducts

Theorem

Suppose that X is a T-functor. Then the following are equivalent:

1. *This functor is a definable set.*
2. *For all sets I, ultrafilters $\mathcal{U}$ on I and models of T, $\mathcal{M}_i$ for $i \in I$, if $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_i$ then*

$$X(\mathcal{M}) = \prod_{\mathcal{U}} X(\mathcal{M}_i).$$

Case study: Property T

Definition

We say that a II_1 factor M has property T if for every $\epsilon > 0$ there is a finite $F \subseteq M$ and $\delta > 0$ such that if H is an M-M correspondence, $\xi \in H$ is a unit vector and $\|[x, \xi]\| \leq \delta$ for all $x \in F$ then there is a central vector $\eta \in H$ such that $\|\eta - \xi\| \leq \epsilon$.

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- Problem 1: We don't have a theory of correspondences.
- Problem 2: We don't have a notion of ultraproduct for correspondences.

The language of correspondences

Fix tracial von Neumann algebras M and N. The language $\mathcal{L}$ of M-N correspondences will include:

- for each $K \in \mathbb{N}$, there will be a sort S_K and for any correspondence H, $S_K(H)$ will be the set of K-bounded vectors. The metric will be induced by the inner product on H;
- for $K < L$ there will be an isometry from S_K to S_L which for a given correspondence will be interpreted as the inclusion map;
- $+$ will be defined on all pairs of sorts and will be interpreted standardly as the restriction of addition from any correspondence; and,
- there will be unary functions for each $c \in M$ and $d \in N$ which implement the left and right actions.

The equivalence of categories

- Let $\mathrm{Corr}(M, N)$ be the category of M-N correspondences with morphisms given by bi-module embeddings.
- For $H \in \mathrm{Corr}(M, N)$, let $\bar{H}$ be the $\mathcal{L}$-structure described on the previous slide and $\mathcal{C}$ be the class of all such structures.
- We want to show two things:
  1. $\mathcal{C}$ is an elementary class, and
  2. the functor $H \to \bar{H}$ is an equivalence of categories.

The main theorem

Theorem

1. *If M and N are tracial von Neumann algebras then the class of M-N correspondences forms an elementary class.*
2. *If M is a II_1 factor then M has property T iff the set of 1-bounded M-central vectors is a definable set for the class of M-M correspondences.*
3. *The class of M-N correspondences is model theoretically very nice: it is stable, classifiable and has a model companion.*

Sketch of the proof of the main theorem

- I will only discuss the closure of correspondences under ultraproducts.
- Suppose H_i is a correspondence for each $i \in I$ and $\mathcal{U}$ is an ultrafilter on I.
- We form the ultraproduct M by taking the ultraproducts of the K-bounded vectors $\prod_{\mathcal{U}} S_K(H_i)$.
- These sets fit together via the embeddings between sorts so that

  $$H^{-} = \bigcup_{K} \left(\prod_{\mathcal{U}} S_K(H_i) \right)$$

  is naturally a complex inner product space on which both M and N act. Taking the closure gives us a correspondence H.

Sketch cont'd

- It is relatively clear that for each K, $\prod_{\mathcal{U}} S_K(H_i) \subseteq S_K(H)$. What we need to see is that this is an equality.
- This comes in two parts: first of all, we show that if $\xi \in H$ is K-bounded in H i.e. is a limit of bounded (but potentially not uniformly so) vectors from H^- then it is in fact the limit of uniformly bounded vectors from H^-.
- Then we show that if ξ is the limit of uniformly bounded vectors then it is the limit of vectors from $\prod_{\mathcal{U}} S_K(H_i)$.

Thank you!