Generalized gauge actions, KMS states, and Hausdorff dimension for higher-rank graphs

Elizabeth Gillaspy
University of Montana

joint with C. Farsi, S. Kang, N. Larsen, J. Packer

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$C^*(E)$ is <u>universal</u> for representations of $\{t_e, t_v\}_{v \in E^0, e \in E^1}$; any collection of partial isometries and projections $\{s_e, s_v\}_{v,e} \subseteq B(\mathcal{H})$ satisfying the above conditions generates a quotient of $C^*(E)$.

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- $E^\infty = \{(e_n)_{n \in \mathbb{N}} : s(e_i) = r(e_{i+1}) \; \forall \; i\}$ is a Cantor set; [IK13] connection between KMS states on $C^*(E)$ and Hausdorff structures on E^∞.
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- [ERRS16] $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$ iff a finite number of moves will convert E into F.

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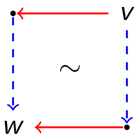

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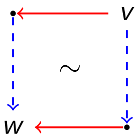
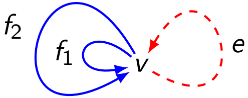


Introduced by Kumjian & Pask in 2000 to give examples of combinatorial, computable C^*-algebras, more general than $C^*(E)$.

Paths in $E \rightsquigarrow k$-dimensional rectangles in Λ.

Example and notation



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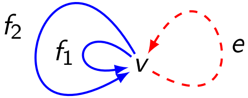

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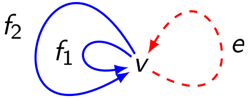

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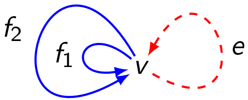

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Note that Λ^0 is the vertices of Λ.

Strongly connected k-graphs

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Theorem ([HLRS15])

If Λ is finite and strongly connected, then the adjacency matrices $\{A_i : 1 \leq i \leq k\} \subseteq M_{\Lambda^0}(\mathbb{N})$,

$$A_i(v, w) = |v\Lambda^{e_i} w| = \#\{\textit{edges of color } i \textit{ from } w \textit{ to } v\}$$

share a unique positive eigenvector $(x_v^\Lambda)_{v \in \Lambda^0}$ of ℓ^1-norm 1.

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Note that $A_iA_j = A_jA_i \;\forall\; i,j$ by the factorization rule.

Infinite paths and Cantor sets

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The collection of sets

$$Z(\lambda) = \{x \in \Lambda^\infty : x = \lambda y\},$$

where $\lambda \in \Lambda$ is a finite path (morphism) in Λ, is a compact open basis for the topology on Λ^∞ making Λ^∞ into a Cantor set.

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$$H^s(Z) = \lim_{\epsilon \to 0} \inf \left\{ \sum_{U_i \in F} (\text{diam } U_i)^s : |F| < \infty, \right.$$
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Moreover, $\exists! \, s \in \mathbb{R} : t < s \Rightarrow H^t(X) = \infty$ and $t > s \Rightarrow H^t(X) = 0$.

We call s the Hausdorff dimension of X.

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Proposition (Farsi-G-Kang-Larsen-Packer)

For any weight functor y on a strongly connected finite k-graph Λ, and any $\beta \geq 0$, the matrices $\{B_i(y, \beta)\}_{1 \leq i \leq k} \in M_{\Lambda^0}$ given by

$$B_i(y, \beta)_{v,w} = \sum_{\lambda \in v\Lambda^{e_i} w} e^{-\beta y(\lambda)}$$

have a unique positive common eigenvector $\xi^{y,\beta}$ of ℓ^1-norm 1.

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Then, for $n = (n_1, \dots, n_k) \in \mathbb{N}^k$, define

$$\rho(B(y,\beta))^n := \rho(B_1(y,\beta))^{n_1} \cdot \rho(B_2(y,\beta))^{n_2} \cdot \cdots \cdot \rho(B_k(y,\beta))^{n_k}.$$

Hausdorff structure and $\mathbb{R}_+$-functors

Theorem (Farsi-G-Kang-Larsen-Packer)

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. For any $\lambda \in \Lambda$, define

$$w_{y,\beta}(\lambda) = e^{-y(\lambda)} \left(\rho(B(y,\beta))^{-d(\lambda)} \xi^{y,\beta}_{s(\lambda)} \right)^{1/\beta}.$$

Suppose moreover that $\rho(B_i(y,\beta)) > \max_{v,w}\{B_i(y,\beta)_{v,w}\}$ for at least one i.

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$$d_{y,\beta}(x,z) := w_{y,\beta}(x \wedge z), \qquad \textit{where } x \wedge z = \max\{\lambda : x, z \in Z(\lambda)\},$$

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is an ultrametric on Λ^∞ which metrizes the cylinder set topology. Also, $(\Lambda^\infty, d_{y,\beta})$ has Hausdorff dimension β and Hausdorff measure

$$\mu_{y,\beta}(Z(\lambda)) = H^\beta(Z(\lambda)) = w_{y,\beta}(\lambda)^\beta = e^{-\beta y(\lambda)} \rho(B(y,\beta))^{-d(\lambda)} \xi^{y,\beta}_{s(\lambda)}.$$

Corollary

For strongly connected finite k-graphs, the authors of [HLRS15] described a measure M on Λ^∞:

$$M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} x^\Lambda_{s(\lambda)},$$

where $\rho(\Lambda) = (\rho(A_1), \rho(A_2), \ldots, \rho(A_k))$, and x^Λ is the common Perron–Frobenius eigenvector of $A_1, \ldots, A_k$.

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Taking $y = 0$ in the previous Theorem gives:

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is an ultrametric on Λ^∞ which metrizes the cylinder set topology. Moreover, the ultrametric spaces $(\Lambda^\infty, d_\beta)$ all have the same Hausdorff measure, namely M.

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Given an $\mathbb{R}_+$-functor on Λ, we obtain an associated action on $C^*(\Lambda)$, and compute the associated KMS states.

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Definition

A positive linear map $\phi : C^*(\Lambda) \to \mathbb{C}$ is a KMS state at (inverse) temperature t for $\alpha^{y,\beta}$ if, for all $\lambda, \eta, \nu, \rho \in \Lambda$,

$$\phi(s_\lambda s_\eta^* s_\nu s_\rho^*) = \phi(\alpha_{it}^{y,\beta}(s_\nu s_\rho^*) s_\lambda s_\eta^*).$$

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

Write $\Phi : C^*(\Lambda) \to C_0(\Lambda^\infty)$ for the usual conditional expectation,

$$\Phi(s_\lambda s_\mu^*) = \begin{cases} \chi_{Z(\lambda)}, & \mu = \lambda \\ 0, & \text{else.} \end{cases}$$

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Theorem (Farsi-G-Kang-Larsen-Packer, [Tho14])

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. Suppose $\rho(B_i(y, \beta)) > 1$ for some $1 \le i \le k$. Then

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When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

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