Matrix Algebras with a Certain Compression Property

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Joint work with Laurent W. Marcoux and Heydar Radjavi

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Assume $\mathcal{A}$ is unital, field is $\mathbb{C}$.

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Also idempotent compressible: $\mathcal{T}_3 := \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$

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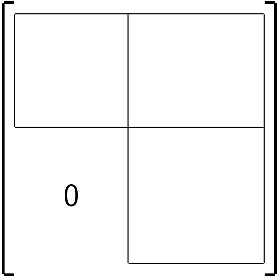

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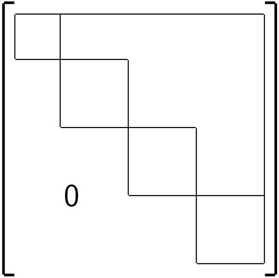

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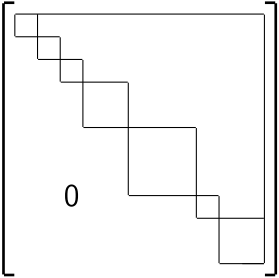

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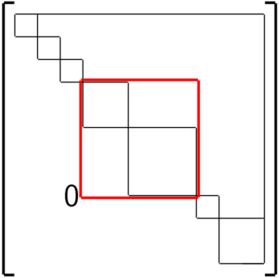

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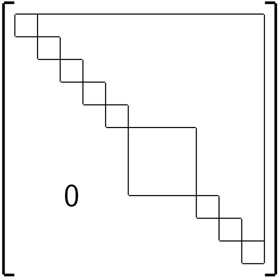

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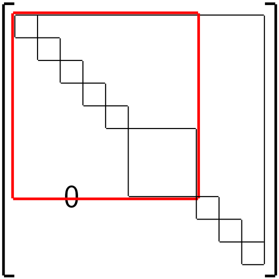

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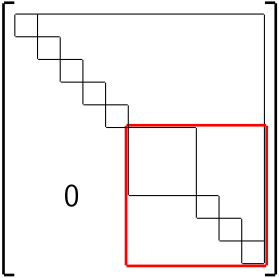

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...but for idempotents, it's no problem!

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THANK YOU!