Recovering locally compact spaces from disjointness relations on function algebras

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Motivation

Theorem (Gelfand-Naimark '43)

If X and Y are compact Hausdorff and $T : C(X) \to C(Y)$ is a $*$-isomorphism, then $X \simeq Y$.

Motivation

Generalizations: We can recover X from $C(X)$ (or $C(X,\mathbb{R})$) as

Banach-Stone '37

A Banach space: $\|f\|_\infty = \sup |f|(X)$.

Gelfand-Kolmogorov '39

A ring: $(f+g)(x) = f(x)+g(x)$, $(fg)(x) = f(x)g(x)$

Milgram '40

A multiplicative semigroup: $(fg)(x) = f(x)g(x)$.

Kaplansky '47

A lattice: $f \leq g \iff \forall x(f(x) \leq g(x))$.

+other recent results, even for non scalar functions.

Motivation

Proof of Gelfand-Naimark

$$\triangleright \qquad \begin{aligned} \text{Open sets} &\longleftrightarrow \text{Closed ideals} \\ U &\longmapsto \{f : f = 0 \text{ outside } U\} \\ &= \overline{\{f : \operatorname{supp}(f) \subseteq U\}} \\ \text{points} &\longleftrightarrow \text{Maximal ideals} \end{aligned} \qquad \triangleleft$$

General proof of the other results

$$\triangleright \qquad \begin{aligned} \text{Open sets} &\longleftrightarrow \text{“ideals”} \\ U &\longmapsto \{f : \operatorname{supp}(f) \subseteq U\} \\ \text{points} &\longleftrightarrow \text{Maximal ideals} \end{aligned} \qquad \triangleleft$$

and Urysohn's Lemma repeatedly.

Goals

- Develop general techniques to recover all these results in full generality;
- Classify isomorphisms for different algebraic structures;
- Non-scalar valued functions;
- Locally compact spaces;
- Non-commutative setting.

Ingredients

- Supports;
- Urysohn's Lemma.

Definitions and notation

Convention:

- $X, Y, \ldots$ will be locally compact Hausdorff (the *domains*).
- $\mathfrak{C}_X, \mathfrak{C}_Y, \ldots$ will be Hausdorff (the *codomains*).
- Let $\theta : X \to \mathfrak{C}_X$ be a fixed continuous function (the *zero*).

Definition

- $[f \neq \theta] = \{x : f(x) \neq \theta(x)\}$;
- $\text{supp}(f) = \overline{[f \neq \theta]}$;
- $C_c(X, \mathfrak{C}_X) = \{f : X \to \mathfrak{C}_X : \text{supp}(f) \text{ is compact}\}$.

Example

If $\mathfrak{C}_X = \mathbb{R}$ (or $\mathbb{C}$), $\theta = 0$.

Definitions and notation

Example

If $\mathfrak{C}_X$ is a group, $\theta = 1$.

Example

If $\mathfrak{C}_X$ is a semigroup with a zero, $\theta = 0$.

Example

If $\mathfrak{C}$ is an lattice with minimum, $\theta = \min \mathfrak{C}$.

Example

If $\mathfrak{C}_X = X$, $\theta = \mathrm{id}_X$.

Definitions and notation

Definition (Urysohn's property)

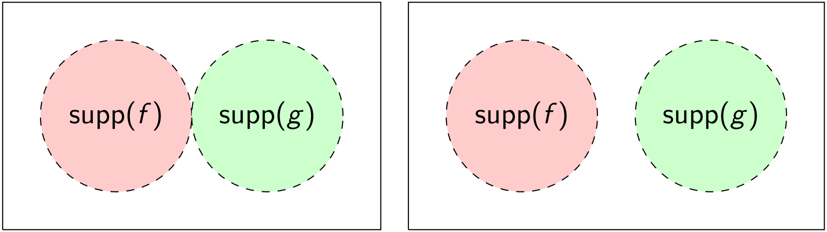
If $\mathcal{A} \subseteq C_c(X, \mathfrak{C}_X)$ is a subset containing θ. We say that $\mathcal{A}$ is *regular* if for all $x \in X$, $U^{\text{open}} \ni x$ and $c \in \mathfrak{C}_X$, there exists $f \in \mathcal{A}$ such that $f(x) = c$ and $\operatorname{supp}(f) \subseteq U$.

Convention: $\mathcal{A}(X)$ will be always regular for some $\mathfrak{C}_X$ and θ.

Some relations

Definition

1. $f \perp g$: if $[f \neq \theta] \cap [g \neq \theta] = \varnothing$ (f and g are *weakly disjoint*);
2. $f \perp\!\!\!\perp g$: if $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$ (f and g are *strongly disjoint*);



$f \perp g$ $\qquad$ $f \perp\!\!\!\perp g$

The main theorem

Theorem

If $T : \mathcal{A}(X) \to \mathcal{A}(Y)$ is a $\perp\!\!\!\perp$-isomorphism then there is a unique homeomorphism $\phi : Y \to X$ such that $\phi(\operatorname{supp} Tf) = \operatorname{supp} f$ for all $f \in \mathcal{A}(X)$.

Proof. Appropriate notion of $\perp\!\!\!\perp$-ideal:

$$U^{\text{open}} \longleftrightarrow \mathrm{I}(U) = \{f \in \mathcal{A} : \operatorname{supp}(f) \subseteq U\}$$

Definition

ϕ is the *T-homeomorphism*.

Why $\perp\!\!\!\perp$ and not $\perp$?

The result is false for $\perp$(weak)-isomorphisms.

Recovering results

Let $\mathbb{C} = \mathbb{R}$, $\theta = 0$.

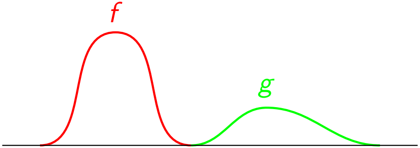
Example (Milgram)

Let $C_c(X, \mathbb{R})$ as a multiplicative semigroup. Then

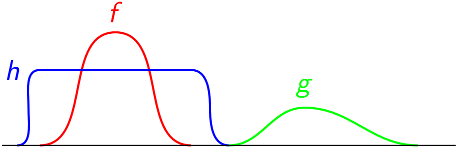
$$f \perp g \iff fg = 0, \text{ the absorbing element}$$

$$f \perp\!\!\!\perp g \iff \exists h(hf = f \text{ and } h \perp g)$$

(same for $\mathbb{C}$, $[0, 1]$, $\mathbb{D}_1, \ldots$).



$f \perp g$



$f \perp\!\!\!\perp g$

Basic $\perp\!\!\!\perp$-isomorphisms

How to classify an "isomorphism" $T : C_c(X, \mathfrak{C}_X) \to C_c(Y, \mathfrak{C}_Y)$?

- $\phi : Y \to X$ homeomorphism: $Tf = f\phi$;
- $\chi : \mathfrak{C}_X \to \mathfrak{C}_Y$ homeo/isomorphism: $Tf = \chi(f\phi)$;
- $\chi : Y \times \mathfrak{C}_X \to \mathfrak{C}_Y$ such that sections $\chi(y, \cdot)$ are homeo/isomorphisms:

$$Tf(y) = \chi(y, f\phi(y)).$$

Definition

- $T : \mathcal{A}(X) \to \mathcal{A}(Y)$ is *basic* if there are such χ and ϕ.
- χ is called the *T-transform*.

Proposition

T is basic iff $f(\phi y) = g(\phi y) \iff Tf(y) = Tg(y)$.

Basic $\perp\!\!\!\perp$-isomorphisms and their transforms

Theorem

$T : C_c(X, \mathfrak{C}_X) \to C_c(Y, \mathfrak{C}_Y)$ is a basic $\perp\!\!\!\perp$-isomorphism, and $\mathfrak{C}_X$ and $\mathfrak{C}_Y$ are "good enough" (e.g. admit some Lie group structure). TFAE

(1) *Each section $\chi(y, \cdot)$ is a continuous;*

(2) *T is continuous with respect to the topologies of pointwise convergence.*

(3) *χ is continuous.*

Proposition

If $\mathfrak{C}_X$ and $\mathfrak{C}_Y$ have some operation $*$ and $\mathcal{A}(X)$ and $\mathcal{A}(Y)$ are (pointwise) $*$-closed, then a basic $T : \mathcal{A}(X) \to \mathcal{A}(Y)$ is a $*$-morphism iff every section $\chi(y, \cdot)$ is a $*$-morphism.

Example of a T-transform

$\mathfrak{C} = \mathbb{R}$, $\theta = 0$.

Example

If $T : C_c(X, \mathbb{R}) \to C_c(Y, \mathbb{R})$ is basic $\perp\!\!\!\perp$-isomorphism

$$Tf(y) = \chi(y, f(\phi(y)))$$

$$\begin{aligned} T \text{ is linear} &\iff \chi(y, \cdot) \text{ is linear for all } y \\ &\iff \chi(y,t) = p(y)t \text{ for some } p(y) \\ &\iff Tf(y) = p(y) f(\phi(y)) \end{aligned}$$

Consequences

For X locally compact Hausdorff, we recover X from $C_c(X,\mathbb{R})$ (or $C_c(X,\mathbb{C})$) from

- Linear $\|\cdot\|_\infty$-isometries (Banach-Stone '37);
- Multiplicative isomorphisms (Milgram '40 $\supseteq$ Gelfand-Kolmogorov '39);
- Lattice isomorphisms (Kaplansky '47);
- Linear $\perp$-preserving isomorphisms (Jarosz '90);
- "Compatibility order"-isomorphisms (Kania-Rmoutil '16).

and for X compact and $C(X)$:

- Linear non-vanishing isomorphisms (Li-Wong '14);
- Non-vanishing group isomorphisms (Hernández-Ródenas '07)

$+$ classifications of isomorphisms for all except Kania-Rmoutil.

New consequences

Endow $C(X, \mathbb{S}^1)$ with the supremum metric:

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

Theorem

If X and Y are Stone spaces and $T : C(X, \mathbb{S}^1) \to C(Y, \mathbb{S}^1)$ is an isometric isomorphism, then there is a homeomorphism $\phi : Y \to X$ and a continuous function $p : Y \to \{\pm 1\}$ such that $Tf(y) = f(\phi y)^{p(y)}$

Rewording

Every isometric isomorphism between unitary groups of commutative unital C*-algebras of real rank zero extends to an isomorphism or conjugate-isomorphism on complementary corners.

Groupoids

A *groupoid* is a small category with inverses.

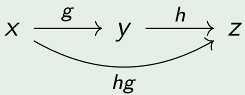
Example

If a group G acts on a set X, the *transformation groupoid*

$$G \ltimes X = \{(gx, g, x) : x \in X, g \in G\}$$

with product

$$(z, h, y)(y, g, x) = (z, hg, x)$$



Groupoid algebras

Discrete case

$\mathbb{C}\mathcal{G} = \langle \delta_g : \delta_g \delta_h = \delta_{gh} \text{ if sensible}, 0 \text{ o/w} \rangle$.

Continuous case: $C_r^*(\mathcal{G})$ is defined in terms of a Haar system. (Renault & Anantharaman-Delaroche)

Example

Jiang-Su algebra $\mathcal{Z}$; Razak-Jacelon algebra $\mathcal{W}$; graph/higher rank graph algebras; Cuntz-Krieger; group algebras; $C(X)$-crossed products; Kirchberg.

Xin Li (arXiv:1802.01190 [math.OA]): classifiable unital stably finite C*-algebras admit (twisted) groupoid models.

New consequences

Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with a regular fully supported Haar system $\lambda_{\mathcal{G}}$ and $\mu_{\mathcal{G}}$ a fully supported regular Borel measure on $\mathcal{G}^{(0)}$. Define the fully supported measure $\lambda_{\mathcal{G}} \otimes \mu_{\mathcal{G}}$ by

$$\int_{\mathcal{G}} f d(\lambda_{\mathcal{G}} \otimes \mu_{\mathcal{G}}) = \int_{\mathcal{G}^{(0)}} \left(\int_{\mathcal{G}^x} f(g) d\lambda_{\mathcal{G}}^x(g) \right) d\mu_{\mathcal{G}}(x)$$

Theorem

If $T : C_c(\mathcal{G}) \to C_c(\mathcal{H})$ is an algebra isomorphism which is $(\lambda_Z \otimes \mu_Z)$-isometric ($Z = \mathcal{G}, \mathcal{H}$), then there are unique topological groupoid isomorphism $\phi : \mathcal{H} \to \mathcal{G}$ and a continuous cocycle $p : \mathcal{H} \to \mathbb{S}^1$ such that

$$Tf(h) = p(h)D(\phi(h))f(\phi(h))$$

where $D(g) = \dfrac{d\lambda_{\mathcal{G}}^{\mathfrak{r}(g)}}{d\left(\phi_* \lambda_{\mathcal{H}}^{\phi^{-1}(\mathfrak{r}(g))}\right)}(g)$, *and in this case* $\mu_G = \phi_* \mu_{\mathcal{H}}$.

New consequences

Similar results hold for:

- Étale Haar groupoids $(\mathcal{G}, \lambda)$, with norm

$$\|f\|_{I,r} = \sup_{x \in \mathcal{G}^{(0)}} \int_{\mathcal{G}^x} |f| d\lambda^x, \qquad f \in C_c(\mathcal{G})$$

  and diagonal $(C_c(\mathcal{G}^{(0)}))$-preserving $\|\cdot\|_{I,r}$-isometric isomorphisms
- For topologically principal, ample groupoids $\mathcal{G}$, and diagonal-preserving isomorphisms of Steinberg algebras $A_R(\mathcal{G})$ over indecomposable rings R.

Thank you.

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