Rigidity in group von Neumann algebra

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Summary

- **Group von Neumann algebras**: definitions; basic properties
- **Classification of group von Neumann algebras**: description of the problems; revisit some older results; (infinite) direct product rigidity; amalgamated free product rigidity; wreath product rigidity; applications to rigidity in $\mathbb{C}^*$-setting
- **Future directions**: some open problems

Group von Neumann algebras

- (Murray-von Neumann '36)
- Γ - countable discrete group

$\rightsquigarrow$ $u : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ - left regular representation

$$u_\gamma(\xi)(\lambda) = \xi(\gamma^{-1}\lambda), \quad \forall \gamma, \lambda \in \Gamma, \xi \in \ell^2\Gamma$$

$\rightsquigarrow$ the von Neumann algebra associated with Γ is

$$\mathcal{L}(\Gamma) := \{u_\gamma \mid \gamma \in \Gamma\}'' = \overline{\mathbb{C}[\Gamma]}^{SOT} \subset \mathfrak{B}(\ell^2\Gamma)$$

$\rightsquigarrow$ $T_i \xrightarrow{SOT} T$ iff $\|T_i\eta - T\eta\| \to 0, \forall \eta \in \ell^2\Gamma$

$\rightsquigarrow$ $\tau(x) = \langle x\delta_e, \delta_e \rangle$ normal, state

- (faithful) $\tau(x^*x) = 0 \Leftrightarrow x = 0$
- (tracial) $\tau(xy) = \tau(yx)$

$\rightsquigarrow$ $\mathcal{L}(\Gamma)$ is a finite von Neumann algebra ($v^*v = 1 \Rightarrow vv^* = 1$)

Group von Neumann Algebras

$\rightsquigarrow$ $\mathcal{L}(\Gamma)$ as algebra of "left convolvers": $\forall\quad \xi, \eta \in \ell^2\Gamma$ define the convolution $\xi * \eta : \Gamma \to \mathbb{C}$

$$\xi * \eta(\gamma) = \sum_{\lambda \in \Gamma} \xi(\gamma\lambda^{-1})\eta(\lambda)$$

$\rightsquigarrow$ $\|\xi * \eta\|_\infty \leq \|\xi\|_2 \|\eta\|_2, \quad \xi * \delta_\gamma = v_{\gamma^{-1}}(\xi), \quad \delta_\gamma * \eta = u_\gamma(\eta)$

$\rightsquigarrow$ $D_\xi = \{\eta \in \ell^2\Gamma \,|\, \xi * \eta \in \ell^2\Gamma\}, \quad D'_\xi = \{\eta \in \ell^2\Gamma \,|\, \eta * \xi \in \ell^2\Gamma\}$, $\rightsquigarrow$ densely def. linear operators

$$\eta \to L_\xi(\eta) = \xi * \eta \quad : D_\xi \to \ell^2\Gamma$$
$$\eta \to R_\xi(\eta) = \eta * \xi \quad : D'_\xi \to \ell^2\Gamma$$

$\rightsquigarrow$ L_ξ, R_ξ have closed graphs and $L_\xi R_\xi = R_\xi L_\xi$

$$lconv(\Gamma) = \{L_\xi \,|\, D_\xi = \ell^2\Gamma\} \subset \mathfrak{B}(\ell^2\Gamma)$$
$$rconv(\Gamma) = \{R_\xi \,|\, D'_\xi = \ell^2\Gamma\} \subset \mathfrak{B}(\ell^2\Gamma)$$

$\rightsquigarrow$ $L_{\xi*\eta} = L_\xi L_\eta, \quad R_{\xi*\eta} = R_\xi R_\eta$

$\rightsquigarrow$ $lconv(\Gamma) = rconv(\Gamma)' = v(\Gamma)' = u(\Gamma)'' = \mathcal{L}(\Gamma)$ (note if $T \in \mathcal{L}(\Gamma)$ then $T = L_\xi$ where $\xi = T\delta_e$)

$\rightsquigarrow$ Fourier expansion $x = \sum_\gamma x_\gamma u_\gamma, \quad x_\gamma = \tau(x u_{\gamma^{-1}})$

Group von Neumann Algebras

Theorem (Murray-von Neumann '36)

$\mathcal{L}(\Gamma)$ *is a* II_1 *factor* ($\mathcal{Z}(\mathcal{L}(\Gamma)) = \mathbb{C}1$) $\Leftrightarrow \forall\, \gamma \neq e$ *we have* $|\gamma^\Gamma| = \infty$, *i.e.* Γ *is icc.*

Examples:

- $\mathbb{F}_n,\ n \geq 2$; $\Gamma_1 * \Gamma_2,\ |\Gamma_1| \geq 2, |\Gamma_2| \geq 3$; $PSL_n(\mathbb{Z}), n \geq 2$;
- $\mathfrak{S}_\infty$; $A \wr \Gamma$, Γ *infinite;*

Major theme of study in von Neumann algebras

- Is it possible a "rigidity theory" in this setting?
- How much information does $\mathcal{L}(\Gamma)$ "remember" of the initial group Γ?
- Is it possible to identify a comprehensive list of canonical algebraic properties of Γ that are recognized by $\mathcal{L}(\Gamma)$?

Classification of group von Neumann algebras

Some non-results:

- (folk) if Γ, Λ infinite abelian then $\mathcal{L}(\Gamma) \cong \mathcal{L}(\Lambda) \cong \mathcal{L}^\infty([0,1])$
- (Connes '76) if Γ, Λ amenable icc then $\mathcal{L}(\Gamma) \cong \mathcal{L}(\Lambda) \cong \overline{\cup_n \mathcal{M}_{2^n}(\mathbb{C})}^{SOT} = \mathcal{R}$ the hyperfinite factor

  Concrete examples: $\mathcal{L}(\mathbb{Z} \wr \mathbb{Z}) \cong \mathcal{L}(\mathbb{Z}_2 \wr \mathbb{Z}) \cong \mathcal{L}(\mathfrak{S}_\infty)$
- (Dykema '93) if Γ_i, Λ_i are infinite amenable then

$$\mathcal{L}(\Gamma_1 * \Gamma_2 * \cdots * \Gamma_n) \cong \mathcal{L}(\Lambda_1 * \Lambda_2 * \cdots * \Lambda_n)$$

- parallel similar results in orbit equivalence, e.g. (Dye '59, Ornstein-Weiss '81, Gaboriau '05)

Conclusion: In general, no memory of the classical group invariants such as torsion, rank, generators and relations, etc

Classification of group of von Neumann algebras

Some results:

- (Murray-von Neumann '43) $\mathcal{L}(\mathbb{F}_2) \ncong \mathcal{L}(\mathfrak{S}_\infty \times \mathbb{F}_2)$
- (McDuff '69) Infinitely many non isomorphic group factors
- (Cowling-Haagerup '89) If $\Gamma < Sp(n,1)$, $\Lambda < Sp(m,1)$ lattices and $n \neq m$ then $\mathcal{L}(\Gamma) \ncong \mathcal{L}(\Lambda)$
- (Ozawa '03) if Γ icc hyperbolic then $\mathcal{L}(\Gamma) \ncong \mathcal{L}(\Lambda \times \Theta)$, for every Λ, Θ infinite groups

Using Popa's deformation/rigidity theory:

- $\mathcal{L}(\Gamma_1 *_\Sigma \Gamma_2 *_\Sigma \cdots *_\Sigma \Gamma_n) \cong \mathcal{L}(\Lambda_1 *_\Omega \Lambda_2 *_\Omega \cdots *_\Omega \Lambda_m)$ implies $n = m$ and there exists $\sigma \in \mathfrak{S}_n$ such that $\mathcal{L}(\Gamma_i) \cong \mathcal{L}(\Lambda_{\sigma_i})$; known when Σ, Ω amenable and Γ_i, Λ_j are
  - infinite property (T) groups, (Ioana-Peterson-Popa '05)
  - non-amenable direct products of infinite groups (C -Houdayer '10)
  - non-amenable containing infinite normal amenable subgroups and Σ, Ω finite (Ioana '12)
- (Ozawa-Popa '07) If $L(\mathbb{F}_n) \cong L(\Lambda)$ then $\forall$ infinite amenable subgroup $\Sigma < \Lambda$ the normalizer $N_\Lambda(\Sigma)$ is amenable
- (Ioana-Popa-Vaes '10) if Γ is non-amenable, $I = \Gamma \wr \mathbb{Z}/\mathbb{Z}$, and $\mathcal{L}(\mathbb{Z}_3 \wr_I (\Gamma \wr \mathbb{Z})) \cong \mathcal{L}(\Lambda)$ then $\mathbb{Z}_3 \wr_I (\Gamma \wr \mathbb{Z}) \cong \Lambda$; other wreath products by (Berbec-Vaes '13)

Unique prime factorization for group von Neumann algebras

Theorem (Ozawa-Popa, '03)

Let Γ_i, Λ_j be icc hyperbolic groups (e.g. $\mathbb{F}_k$, $k \geq 2$). If

$$\mathcal{L}(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n) \cong \mathcal{L}(\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_m)$$

then $n = m$ and there exist scalars $t_1 t_2 \cdots t_n = 1$ and a permutation $\sigma \in \mathfrak{S}_n$ such that

$$\mathcal{L}(\Gamma_i)^{t_i} \cong \mathcal{L}(\Lambda_{\sigma_i}), \text{ for all } 1 \leq i \leq n$$

- the result still holds for all icc biexact groups; includes:
  - all groups hyperbolic relative to families of amenable subgroups
  - $\mathbb{Z} \wr \Gamma$, for every Γ non-elementary hyperbolic
  - $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$
- we cannot do better at the factors level; by Voiculescu's formula $\mathcal{L}(\mathbb{F}_5 \times \mathbb{F}_3) \cong \mathcal{L}(\mathbb{F}_2 \times \mathbb{F}_9)$
- this parallels results of (Monod-Shalom '06) in orbit equivalence

Direct product rigidity

Theorem (C - De Santiago-Sinclair, '15)

Let Γ_i be icc hyperbolic groups (or more generally biexact) and Λ be an arbitrary group. If

$$\mathcal{L}(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n) \cong \mathcal{L}(\Lambda)$$

then $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$ and there exist scalars $t_1 t_2 \cdots t_n = 1$ such that

$$\mathcal{L}(\Gamma_i)^{t_i} \cong \mathcal{L}(\Lambda_i), \text{ for all } 1 \leq i \leq n$$

- a contrast point with the OE counterpart (Monod-Shalom '06) is no need to assume strong forms of ergodicity on the "target data"

Corollary

Let Γ_i be icc biexact groups and Λ be an arbitrary group such that

$$\mathbb{C}^*_r(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n) \cong \mathbb{C}^*_r(\Lambda).$$

Then $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$ where Λ_i are icc, biexact groups.

Infinite direct product rigidity

- Unique prime factorization results for infinite tensor products of solid factors were obtained by (Isono '16); motivated by these results we can ask the following basic question:
Can we get a similar product rigidity result for infinite direct sum groups?
- **Canonical obstruction**:

$$\begin{aligned}\mathcal{L}(\oplus_{i\in\mathbb{N}}\Gamma_i) &\cong \bar{\otimes}_{i\in\mathbb{N}}\mathcal{L}(\Gamma_i)\\ &\cong \bar{\otimes}_{i\in\mathbb{N}}(\mathcal{L}(\Gamma_i)^{1/2}\otimes\mathcal{M}_2(\mathbb{C}))\\ &\cong (\bar{\otimes}_{i\in\mathbb{N}}\mathcal{L}(\Gamma_i)^{1/2})\bar{\otimes}\mathcal{R}\\ &\cong (\bar{\otimes}_{i\in\mathbb{N}}\mathcal{L}(\Gamma_i)^{1/2})\bar{\otimes}\mathcal{R}\bar{\otimes}\mathcal{R}\\ &\cong (\bar{\otimes}_{i\in\mathbb{N}}\mathcal{L}(\Gamma_i))\bar{\otimes}\mathcal{R}\\ &\cong \mathcal{L}(\oplus_{i\in\mathbb{N}}\Gamma_i\oplus A),\end{aligned}$$

for any A icc amenable group.

Infinite direct product rigidity

Theorem (C - Udrea, '18)

Let $(\Gamma_i)_{i\in\mathbb{N}}$ be icc, biexact, property (T) groups and let Λ be an arbitrary group such that

$$\mathcal{L}(\Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n \oplus \cdots) \cong \mathcal{L}(\Lambda).$$

Then $\Lambda = (\Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_n \oplus \cdots) \oplus A$ where A is icc amenable. Also there are positive scalars $t_1, t_2, ..., t_n, ...$ so that for each $k \in \mathbb{N}$ we have

$$\mathcal{L}(\Gamma_1)^{t_1} \cong \mathcal{L}(\Lambda_1), \quad \mathcal{L}(\Gamma_2)^{t_2} \cong \mathcal{L}(\Lambda_2), \quad \ldots \quad \mathcal{L}(\Gamma_k)^{t_k} \cong \mathcal{L}(\Lambda_k),$$
$$\mathcal{L}(\Gamma_{k+1} \oplus \Gamma_{k+2} \oplus \cdots) \cong \mathcal{L}((\Lambda_{k+1} \oplus \Lambda_{k+2} \oplus \cdots) \oplus A)$$

- the result applies to all Γ_i's
  - uniform lattices in $Sp(n,1)$, $n \geq 2$
  - Gromov random groups with density $3^{-1} < d < 2^{-1}$
  - prop (T), hyperbolic relative to finitely generated amenable groups constructed by (Arzhantseva-Minasyan-Osin '07)

Infinite direct product rigidity in $\mathbb{C}^*$-setting

Corollary

Let $(\Gamma_i)_{i\in\mathbb{N}}$ be icc, biexact, property (T) groups and let Λ be an arbitrary group such that

$$\mathbb{C}_r^*(\Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n \oplus \cdots) \cong \mathbb{C}_r^*(\Lambda).$$

Then $\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_n \oplus \cdots$ where Λ_i are icc, biexact, property (T) groups.

Proof. $\oplus_{i\in\mathbb{N}}\Gamma_i$ has trivial amenable radical and by (Breuillard-Kalantar-Kennedy-Ozawa '14) $\mathbb{C}_r^*(\oplus_{i\in\mathbb{N}}\Gamma_i)$ has unique trace; thus the isomorphism $\mathbb{C}_r^*(\oplus_{i\in\mathbb{N}}\Gamma_i) \cong \mathbb{C}_r^*(\Lambda)$ preserves the trace and hence lifts to the von Neumann algebras level and it follows from the previous theorem that $\Lambda = \oplus_{i\in\mathbb{N}}\Lambda_i \oplus A$; however by uniqueness of the trace again (Breuillard-Kalantar-Kennedy-Ozawa '14) implies that $A = 1$ □

Amalgamated free product rigidity

Theorem (C - Ioana '17)

Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ *be an amalgam satisfying the following:*

- $\Gamma_1 = \Gamma_1^1 \times \Gamma_1^2$, $\Gamma_2 = \Gamma_2^1 \times \Gamma_2^2$ *where* Γ_i^j, *are icc, non-amenable, biexact groups;*
- Σ *is icc amenable and* $[\Sigma : \gamma\Sigma\gamma^{-1} \cap \Sigma] = \infty$ *for every* $\gamma \in \Gamma_i \setminus \Sigma$.

Assume Λ *is an arbitrary group such that*

$$\mathcal{L}(\Gamma_1 *_\Sigma \Gamma_2) = \mathcal{L}(\Lambda).$$

Then $\Lambda = \Lambda_1 *_\Delta \Lambda_2$ *and there exists* $u \in \mathcal{U}(\mathcal{L}(\Gamma))$ *s.t.*

$$\mathcal{L}(\Lambda_1) = u\mathcal{L}(\Gamma_1)u^*, \quad \mathcal{L}(\Lambda_2) = u\mathcal{L}(\Gamma_2)u^*, \quad \mathcal{L}(\Delta) = u\mathcal{L}(\Sigma)u^*.$$

Examples:

1. $[(H_1 * \Theta) \times (K_1 * \Omega)] *_{(\Theta\times\Omega)} [(H_2 * \Theta) \times (K_2 * \Omega)]$ $\forall H_i, K_i-$ *biexact*, $\Theta, \Omega-$ *icc, amenable*
2. $[(A \wr H) \times (A \wr H)] *_{(A\wr C \times A\wr C)} [(A \wr H) \times (A \wr H)]$ $\forall H$ *hyperbolic*, $C < H$ *max. inf. amenable*

W^*-superrigidity

Theorem (C - Ioana '17)

Let $\Sigma < \Gamma_0$ be groups satisfying the following

- Γ_0 *be icc, non-amenable, biexact;*
- Σ *is icc, amenable, and satisfies* $[\Sigma : \Sigma \cap \gamma\Sigma\gamma^{-1}] = \infty$ *for every* $\gamma \in \Gamma_0 \setminus \Sigma$;
- *the centralizer in* Γ_0 *of any finite index subgroup of* $\Sigma \cap \gamma\Sigma\gamma^{-1}$ *is trivial, for all* $\gamma \in \Gamma_0$.

Let $\Gamma := (\Gamma_0 \times \Gamma_0) *_{diag(\Sigma)} (\Gamma_0 \times \Gamma_0)$.
If Λ *is an any group and* $\theta : \mathcal{L}(\Gamma) \to \mathcal{L}(\Lambda)$ *is any* $*$*-isomorphism then there exist a group isomorphism* $\delta : \Gamma \to \Lambda$ *a unitary* $u \in \mathcal{L}(\Lambda)$*, and a character* $\eta : \Gamma \to \mathbb{T}$ *such that*

$$\theta(u_\gamma) = \eta(\gamma) u v_{\delta(\gamma)} u^*, \quad \forall \gamma \in \Gamma.$$

Examples

$[(\mathfrak{S}_\infty \wr \mathbb{F}_n) \times (\mathfrak{S}_\infty \wr \mathbb{F}_n)] *_{diag(\mathfrak{S}_\infty \wr \mathbb{Z})} [(\mathfrak{S}_\infty \wr \mathbb{F}_n) \times (\mathfrak{S}_\infty \wr \mathbb{F}_n)] \qquad n \geq 2$

Applications to $\mathbb{C}^*$-superrigidity

Corollary (C - Ioana '17)

Let $\Sigma < \Gamma_0$ be groups satisfying the following

- Γ_0 *be icc, non-amenable, biexact;*
- Σ *is icc, amenable, and satisfies* $[\Sigma : \Sigma \cap \gamma\Sigma\gamma^{-1}] = \infty$ *for every* $\gamma \in \Gamma_0 \setminus \Sigma$;
- *the centralizer in* Γ_0 *of any finite index subgroup of* $\Sigma \cap \gamma\Sigma\gamma^{-1}$ *is trivial, for all* $\gamma \in \Gamma_0$.

Let $\Gamma := (\Gamma_0 \times \Gamma_0) *_{diag(\Sigma)} (\Gamma_0 \times \Gamma_0)$.

If Λ *is an any group and* $\theta : \mathbb{C}^*_r(\Gamma) \to \mathbb{C}^*_r(\Lambda)$ *is any* $*$*-isomorphism then there exist a group isomorphism* $\delta : \Gamma \to \Lambda$ *a unitary* $u \in \mathcal{L}(\Lambda)$, *and a character* $\eta : \Gamma \to \mathbb{T}$ *such that*

$$\theta(u_\gamma) = \eta(\gamma) u v_{\delta(\gamma)} u^*, \quad \forall \gamma \in \Gamma.$$

• this provide the first examples of non-amenable $\mathbb{C}^*$-superrigid groups; the only other known examples by (Scheinberg '74, Knuby-Raum-Thiel-White '17, Eckhart-Raum '18, Omland '18)

Wreath product rigidity

Theorem (C - Udrea '18)

- *Let* H, Γ_1, Γ_2 *icc, biexact, property (T) groups;*
- $\Gamma = \Gamma_1 \times \Gamma_2$.

Let Λ *be an arbitrary group and let* $\theta : \mathcal{L}(H \wr \Gamma) \to \mathcal{L}(\Lambda)$ *be a* $*$*-isomorphism. Then*

- Σ, Ψ *icc, property (T) groups,* A *icc amenable, an action* $\Psi \curvearrowright^{\alpha} A$ *such that* $\Lambda = (\Sigma^{(\Psi)} \oplus A) \rtimes_{\beta \oplus \alpha} \Psi$, *where* $\Psi \curvearrowright^{\beta} \Sigma^{(\Psi)}$ *is the Bernoulli action.*

Also there is a group isomorphism $\delta : \Gamma \to \Psi$, *a character* $\eta : \Gamma \to \mathbb{T}$, *a* $*$*-isomorphism* $\theta_0 : \mathcal{L}(H^{(\Gamma)}) \to \mathcal{L}(\Sigma^{(\Psi)} \oplus A)$, *and a unitary* $\nu \in \mathcal{L}(\Lambda)$ *such that for all* $x \in \mathcal{L}(H^{(\Gamma)})$, $\gamma \in \Gamma$ *we have*

$$\theta(x\gamma) = \eta(\gamma)\nu\theta_0(x)\delta(\gamma)\nu^*.$$

Corollary (C - Udrea '18)

Let H, Γ_1, Γ_2 *be icc, biexact, property (T) groups and let* $\Gamma = \Gamma_1 \times \Gamma_2$. *If* Λ *is an arbitrary group so that* $\mathbb{C}^*_r(H \wr \Gamma) = \mathbb{C}^*_r(\Lambda)$ *then* $\Lambda = \Sigma \wr \Gamma$, *where* Σ *is an icc, biexact, property (T) group.*

Ideas behind the proof of the infinite direct product rigidity here

Questions

Open Problem

Does the infinite product rigidity still holds if one removes the property (T) assumption (i.e. for infinite direct sum of icc non-amenable biexact groups)?

Open Problem

Are there any instances when the plain free product rigidity holds?

$L(\Gamma_1 * \Gamma_2) = L(\Lambda) \overset{?}{\Rightarrow} \Lambda = \Lambda_1 * \Lambda_2$ and $L(\Lambda_i) \cong L(\Gamma_i)$ for all $i = 1, 2$.

Open Problem

Find other algebraic features reconstructible from von Neumann algebras. In particular, is there an HNN-extension group that can be reconstructed from its von Neumann algebra?

Open Problem

Is there a hyperbolic group Γ s.t. whenever Λ is a group satisfying $\mathcal{L}(\Gamma) \cong \mathcal{L}(\Lambda)$ it follows that Λ is hyperbolic as well?

THANK YOU!

supplemental2. **Step 1:** let $\{\gamma : \gamma \in \Gamma\}'' = \mathcal{L}(\oplus_{i\in I}\Gamma_i) = M = \mathcal{L}(\Lambda) = \{\lambda : \lambda \in \Lambda\}''$

$\rightsquigarrow$ (Popa-Vaes) co-multiplication along Λ i.e. $\Delta : M \to M\bar{\otimes}M$ given by $\Delta(\lambda) = \lambda \otimes \lambda$

$\rightsquigarrow$ $\forall i, j \in I$ we have $\Delta(\mathcal{L}(\Gamma_{I\setminus\{i\}})), \Delta(\mathcal{L}(\Gamma_i)) \subseteq \Delta(M) \subseteq M\bar{\otimes}M = M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}} \oplus \Gamma_j)$ using (Popa-Vaes '12) control of normalizers $\Rightarrow$

$$\Delta(\mathcal{L}(\Gamma_{I\setminus\{i\}})) \prec M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}}) \quad \text{or} \quad \Delta(\mathcal{L}(\Gamma_i)) \prec M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}})$$

However, if $\Delta(\mathcal{L}(\Gamma_i)) \prec M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}})$ for all j then $\Delta(\mathcal{L}(\Gamma_i)) \prec M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus F})$ for all $F \subset I$ finite; it follows that $\Delta(\mathcal{L}(\Gamma_i))$ is amen. rel. to $M\bar{\otimes}1$ forcing Γ_i amenable, contradiction; hence $\forall i, \exists j$ so that

$$\Delta(\mathcal{L}(\Gamma_{I\setminus\{i\}})) \prec M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}})$$

Ideas behind the proof — Infinite direct product rigidity

supplemental3. **Step 2:** $\|E_{M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}})}(\Delta(u))\|_2 \geq C > 0$ for all $u \in \mathcal{U}(L(\Gamma_{I\setminus\{i\}}))$

⇝ (ultrapower tech Ioana '11) let $\mathcal{G} = \{\Sigma : \Sigma \leqslant \Lambda\}$ and $\mathcal{J}$ the directed set of all small sets over $\mathcal{G}$

⇝ if $\mathcal{L}(\Gamma_{I\setminus\{i\}}) \nprec \mathcal{L}(\Sigma)$ for all $\Sigma \in \mathcal{G}$ then for each $S \in \mathcal{J}$ there exists $\lambda_S \in \Lambda \setminus S$ such that

$$\|E_{\mathcal{L}(\Gamma_{I\setminus\{j\}})}(\lambda_S)\|_2 \geq C/2$$

⇝ if ω is cofinal ultrafilter on $\mathcal{J}$ then $E_{\mathcal{L}(\Gamma_{I\setminus\{j\}})^\omega}(\lambda^\omega) \neq 0$ where $\lambda^\omega = (\lambda_S)_S$

⇝ Assume: $\lambda^\omega \in \mathcal{L}(\Gamma_{I\setminus\{j\}})^\omega \subseteq \mathcal{L}(\Gamma_j)' \cap M^\omega \Rightarrow \mathcal{L}(\Gamma_j) \subseteq \lambda^\omega M(\lambda^\omega)^{-1} \cap M = \mathcal{L}(\lambda^\omega\Lambda(\lambda^\omega)^{-1} \cap \Lambda)$

⇝ $\lambda^\omega\Lambda(\lambda^\omega)^{-1} \cap \Lambda = \cup_n C(\Sigma_n)$ where $\Sigma_n \notin \mathcal{G}$, descending

⇝ for all $\mathcal{G}$ either $\mathcal{L}(\Gamma_{I\setminus\{i\}}) \prec \mathcal{L}(\Sigma)$ for some $\Sigma \in \mathcal{G}$ or $\mathcal{L}(\Gamma_j) \prec \mathcal{L}(\cup_n C(\Sigma_n))$ with $\Sigma_n \notin \mathcal{G}$

Ideas behind the proof — Infinite direct product rigidity

supplemental4. **Step 3:** $\exists\, \Sigma \leqslant \Lambda$ s. t. $\mathcal{L}(\Gamma_{I\setminus\{i\}}) \prec \mathcal{L}(\Sigma)$ with $C(\Sigma)$ is non-amenable;

$\rightsquigarrow$ Assume: $\mathcal{L}(\Gamma_{I\setminus\{i\}}) \subseteq \mathcal{L}(\Sigma)$; splitting prop of tensors $\Rightarrow \mathcal{L}(\Gamma_{I\setminus\{i\}})\bar{\otimes}B = \mathcal{L}(\Sigma)$ with $B \subseteq \mathcal{L}(\Gamma_i)$

$\rightsquigarrow$ solidity of $\mathcal{L}(\Gamma_i)$ imply B is atomic, so up to corners we can assume

$$\mathcal{L}(\Gamma_{I\setminus\{i\}}) = \mathcal{L}(\Sigma)$$

$\rightsquigarrow$ passing to relative commutants we get

$$\mathcal{L}(\Gamma_i) = \mathcal{L}(\Sigma)' \cap \mathcal{L}(\Lambda) \subseteq \mathcal{L}(vC(\Sigma))$$

where $vC(\Sigma) = \{\lambda \in \Lambda \,:\, |\lambda^{\Sigma}| < \infty\}$—virtual centralizer of Σ

$\rightsquigarrow$ let $\mathcal{O}_1, \mathcal{O}_2, \ldots$ all finite orbits under Σ-conjug.; let $\Omega_k = \langle \mathcal{O}_1, \ldots, \mathcal{O}_k \rangle$ and note $\cup_k \Omega_k = vC(\Sigma)$

$\rightsquigarrow$ using property (T) we get

$$\mathcal{L}(\Gamma_i) = \mathcal{L}(\Sigma)' \cap \mathcal{L}(\Lambda) \subseteq \mathcal{L}(\Omega_\ell)$$

hence $\mathcal{L}(\Gamma_{I\setminus\{i\}})\bar{\otimes}\mathcal{L}(\Gamma_i) = \mathcal{L}(\Sigma \vee \Omega_\ell) = \mathcal{L}(\Lambda)$; $\Rightarrow \Sigma \vee \Omega_\ell = \Lambda$ and $vC(\Sigma) = \Omega_\ell$

$\rightsquigarrow$ as Σ has finite index subgroup commuting with $vC(\Sigma)$ then Λ is commensurable to a product group

Ideas behind the proof — Infinite direct product rigidity

main. **Step 4:** "perturbing" Σ and $vC(\Sigma)$ up to finite index and using (Ozawa-Popa '03) we get that $\Lambda = \Sigma_i \oplus vC(\Sigma_i)$ and $t_i > 0$ such that

$$\mathcal{L}(\Gamma_{I\setminus\{i\}})^{t_i} = \mathcal{L}(\Sigma_i) \text{ and there is } \mathcal{L}(\Gamma_i)^{1/t_i} = \mathcal{L}(vC(\Sigma_i))$$

$\rightsquigarrow$ using these product decompositions for every i one derive the desired conclusion

• since the "equalities" above are up to corners and partial conjugacy all the statements above are virtual (i.e. up to finite index/intersection) and their proofs are pretty involved technically; to get to a honest direct product we often have to use factoriality of the original algebras.