Continuity and Holomorphicity of Symbols of Weighted Composition Operators

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$\mathcal{F}(X)$ - the space of all complex-valued functions over a set X with the pointwise topology.

A *Banach space of functions* (BSF) over X is a linear subspace $\mathbf{F} \subset \mathcal{F}(X)$ with a complete norm, such that the inclusion

$$J_{\mathbf{F}} : \mathbf{F} \to \mathcal{F}(X)$$

is continuous.

$\omega : X \to \mathbb{C}$ is a *multiplier* of a BSF $\mathbf{F}$ if $\omega f \in \mathbf{F}$, for every $f \in \mathbf{F}$.

> If X is a topological space (domain in $\mathbb{C}^n$), and $\mathbf{F}$ consists of continuous (holomorphic) functions, are all multipliers of $\mathbf{F}$ continuous (holomorphic)?

Plan

- Banach Spaces of Continuous Functions
- Basics on Weighted Composition Operators
- The Main Question and Results (Continuous Case)
- Holomorphic Case

Banach Spaces Of Continuous Functions

X - locally compact Hausdorff topological space.

$\mathcal{C}(X)$ - the space of all continuous complex-valued functions over X with the compact-open topology.

A *Banach space of continuous functions* (BSCF) over X is BSF contained in $\mathcal{C}(X)$.

If $\mathbf{F}$ is a BSCF, then the inclusion

$$J_{\mathbf{F}} : \mathbf{F} \to \mathcal{C}(X)$$

is continuous.

$\mathbf{F}$ is *compactly embedded* (in $\mathcal{C}(X)$) if $J_{\mathbf{F}}$ is a compact operator.

Example 1

- If d is a metric on X compatible with the topology, then $Lip(X, d)$ is a compactly embedded BSCF.
- If X is non-discrete, then the space $\mathcal{C}_b(X)$ of bounded continuous functions with the supremum-norm is a non-compactly embedded BSCF.
- Banach spaces of *holomorphic* functions are always compactly embedded (Montel's theorem).

A collection $\mathbf{F} \subset \mathcal{C}(X)$ *generates the topology* of X if the topology of X is the minimal topology in which all elements of $\mathbf{F}$ are continuous.

Example 2

- If $X \subset \mathbb{C}^n$, the coordinate functions generate the topology of X.
- The function $t \to e^{it}$ does NOT generate the topology of $[0, 2\pi)$.

Point Evaluations

For a BSCF $\mathbf{F}$ define the *evaluation map* $\kappa_{\mathbf{F}} : X \to \mathbf{F}^*$ by

$$\langle f, \kappa_{\mathbf{F}}(x)\rangle = f(x).$$

Then $\kappa_{\mathbf{F}}(x)$ is the *point evaluation* at x on $\mathbf{F}$.

Proposition 1

(i) *$\kappa_{\mathbf{F}}$ is an injection if and only if $\mathbf{F}$ separates points of X, i.e. if $x, y \in X$ and $x \neq y$, there is $f \in \mathbf{F}$ with $f(x) \neq f(y)$.*

(ii) *$\kappa_{\mathbf{F}}$ is weak* continuous.*

(iii) *$\kappa_{\mathbf{F}}$ is a weak* topological embedding (i.e. a homeomorphism onto its image) if and only if $\mathbf{F}$ generates the topology of X.*

(iv) *(J. Wada, 1961) $\kappa_{\mathbf{F}}$ is norm continuous if and only if $\mathbf{F}$ is compactly embedded.*

Weighted Composition Operators

Let X and Y be locally compact. Let $\Phi : Y \to X$ and $\omega : Y \to \mathbb{C}$.

A *weighted composition operator* (WCO) with *composition symbol* Φ and *multiplicative symbol* ω is a linear map $W_{\omega,\Phi}$ from $\mathcal{F}(X)$ into $\mathcal{F}(Y)$ defined by

$$[W_{\omega,\Phi}f](y) = \omega(y) f(\Phi(y)).$$

Let $\mathbf{F}$, $\mathbf{E}$ be BSCF's over X and Y.

If $W_{\omega,\Phi}\mathbf{F} \subset \mathbf{E}$, then $W_{\omega,\Phi} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$, due to Closed Graph theorem.

Hence, $W_{\omega,\Phi}$ is either bounded from $\mathbf{F}$ into $\mathbf{E}$, or not well-defined from $\mathbf{F}$ into $\mathbf{E}$.

It is quite difficult to determine in general which WCO's are bounded from given $\mathbf{F}$ into given $\mathbf{E}$.

Special cases:

- if $\omega \equiv 1$, then $W_{\omega,\Phi} = C_\Phi$ is a *composition operator* with symbol Φ.
- if $X{=}Y$ and $\Phi = Id$, then $W_{\omega,\Phi} = M_\omega$ is a *multiplication* operator with weight ω.

If both C_Φ and M_ω are bounded, then $W_{\omega,\Phi} = M_\omega C_\Phi$, but it is possible that $W_{\omega,\Phi}$ is bounded, while both M_ω and C_Φ are NOT bounded.

WCO's are often the only isometric isomorphisms between important BSCF's (Banach-Stone-type theorems):

- $\mathcal{C}(X)$ for compact X;
- $\mathcal{C}_0(X)$;
- Hardy Space H^p, $p \neq 2$ (Forelli, 1964);
- Bergman Space $L^p_a(\mathbb{D})$, $p \neq 2$ (Kolaski, 1981).

The Main Question

Given that $W_{\omega,\Phi} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$, when can we guarantee continuity of ω and Φ?

Some obstructions:

- If $\omega(x) = 0$, then $W_{\omega,\Phi}$ does not depend on $\Phi(x)$.
- We cannot reconstruct Φ and ω from $W_{\omega,\Phi}$ if there are $x, y \in X$ such that $\kappa_{\mathbf{F}}(x)$ and $\kappa_{\mathbf{F}}(y)$ are linearly dependent.

$\mathbf{F}$ is 2-*independent* if $\kappa_{\mathbf{F}}(x)$ and $\kappa_{\mathbf{F}}(y)$ are linearly independent for $x \neq y$, i.e. there are $f, g \in \mathbf{F}$, such that $f(x) = 1$, $f(y) = 0$, $g(x) = 0$ and $g(y) = 1$.

Example 3

Assume that $X \subset \mathbb{C}$ and $1, Id \in \mathbf{F}$. Assume also that ω does not vanish. Then $W_{\omega,\Phi}\mathbf{F} \subset \mathbf{E}$ leads to $\omega = W_{\omega,\Phi}1 \in \mathbf{E} \subset \mathcal{C}(Y)$ and $\omega\Phi = W_{\omega,\Phi}Id \in \mathbf{E} \subset \mathcal{C}(Y)$. Hence, $\Phi \in \mathcal{C}(Y)$.

Example 4

Let $X = [0, 2\pi)$. Let $K : X \times X \to \mathbb{C}$ be defined by

$$K(x, y) = \frac{1}{1 - \frac{1}{4} e^{i(x-y)}}.$$

K is a continuous positive definite complete Nevanlinna-Pick kernel.

The RKHS $\mathbf{F}_K$ defined by K is a compactly embedded and 2-independent BSCF.

$\Phi : X \to X$ is defined by

$$\Phi(x) = x + \pi \pmod{2\pi}.$$

Φ is not continuous on X, but C_Φ is a unitary operator on $\mathbf{F}_K$.

Note that $\mathbf{F}_K$ does NOT generate the topology of X.

Proposition 2 (B.)

(i) *If* $\mathbf{F}$ *is* 2*-independent,* $W_{\omega,\Phi} \in \mathcal{L}(\mathbf{F},\mathbf{E})$ *and* Φ *is continuous, then* ω *is continuous.*

(ii) *If* $\mathbf{F}$ *generates the topology of* X*,* $W_{\omega,\Phi} \in \mathcal{L}(\mathbf{F},\mathbf{E})$ *and* ω *is continuous and non-vanishing, then* Φ *is continuous.*

Proof of (ii).

$W^*_{\omega,\Phi}\kappa_{\mathbf{E}}(y) = \omega(y) \cdot \kappa_{\mathbf{F}}(\Phi(y))$. Hence, $\Phi = \kappa_{\mathbf{F}}^{-1} \circ \left(\frac{1}{\omega} \cdot W^*_{\omega,\Phi}\kappa_{\mathbf{E}}\right)$. □

Immediate consequences:

- If $X = Y$, $\mathbf{F}$ is 2-independent, and $M_\omega \in \mathcal{L}(\mathbf{F},\mathbf{E})$, then ω is continuous.
- If $\mathbf{F}$ generates the topology of X and $C_\Phi \in \mathcal{L}(\mathbf{F},\mathbf{E})$, then Φ is continuous.
- If $\mathbf{F}$ generates the topology of X, $1 \in \mathbf{F}$, $W_{\omega,\Phi} \in \mathcal{L}(\mathbf{F},\mathbf{E})$ and ω is non-vanishing, then ω and Φ are continuous.

Example 5

Let $X = [0, 2\pi)$. Let $L : X \times X \to \mathbb{C}$ be defined by

$$L(x, y) = \frac{(x + \pi)(y + \pi)}{1 - \frac{1}{4} e^{i(x-y)}}.$$

L is a continuous positive definite complete Nevanlinna-Pick kernel. The RKHS $\mathbf{F}_L$ defined by L is a compactly embedded and 2-independent BSCF.

The "coordinate" function $x + \pi$ belongs to $\mathbf{F}_L$, and so $\mathbf{F}_L$ generates the topology of X.

Define $\Phi : X \to X$ and $\omega : X \to \left[\frac{1}{3}, 3\right]$ by

$$\Phi(x) = x + \pi \pmod{2\pi}, \quad \omega(x) = \frac{x + \pi}{\pi + \Phi(x)}.$$

Φ and ω are not continuous, but $W_{\omega,\Phi}$ is a unitary operator on $\mathbf{F}_L$.

The Main Result

The key factor is how $\kappa_{\mathbf{F}}(X)$ sits in $(\mathbf{F}^*, \textit{weak}^*)$.

Instead of demanding that $1 \in \mathbf{F}$ and $\mathbf{F}$ generates the topology we will impose conditions which are less restrictive or easier to verify:

- topological properties of X;
- properties of elements of $\mathbf{F}$;
- norm of point evaluations.

Theorem 1 (B.)

*If $\mathbf{F}$ is 2-independent, $W_{\omega,\Phi} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$ and ω is non-vanishing, then ω and Φ are continuous whenever **either** of the following holds:*

(i) *X is compact;*

(ii) *ω is bounded and $\mathbf{F} \subset \mathcal{C}_0(X)$;*

(iii) *$\mathbf{F}$ and $\mathbf{E}$ are compactly embedded, bounded functions form a dense set in $\mathbf{F}$, and $\lim\limits_{x\to\infty} \|\kappa_{\mathbf{F}}(x)\| = +\infty$.*

Idea of the Proof

Note: (i)$\Rightarrow\kappa_{\mathbf{F}}(X)$ is weak* closed, (ii)$\Rightarrow\kappa_{\mathbf{F}}(X)\cup\{0_{\mathbf{F}^*}\}$ is weak* closed, and (iii)$\Rightarrow$ $\{Normalized\ point\ evaluations\}\cup\{0_{\mathbf{F}^*}\}$ is weak* closed.

Lemma 1

Let H be a topological vector space, with the operation of scalar multiplication given by $\mu:\mathbb{C}\times H\to H$, and let $K\subset H$ be bounded and contain no pairs of linearly dependent elements. Then:

(i) *If K is closed, then $\mu\left|_{\mathbb{C}\backslash\{0\}\times K}\right.$ is a homeomorphism onto its image;*

(ii) *If $K\cup\{0_H\}$ is closed, then $\mu|_{B\times K}$ is a homeomorphism onto its image, for every bounded $B\subset\mathbb{C}\backslash\{0\}$.*

If $W_{\omega,\Phi}\in\mathcal{L}(\mathbf{F},\mathbf{E})$, then $\mu(\omega(y),\kappa_{\mathbf{F}}(\Phi(y)))=W^*_{\omega,\Phi}\kappa_{\mathbf{E}}(y)$, for $y\in Y$. In the simplest case when $\kappa_{\mathbf{F}}(X)$ is weak* closed,

$$\omega=proj_1\circ\left(\mu\left|_{\mathbb{C}\backslash\{0\}\times\kappa_{\mathbf{F}}(X)}\right.\right)^{-1}\circ W^*_{\omega,\Phi}\circ\kappa_{\mathbf{E}}$$

is continuous. Continuity of Φ and other cases are done similarly.

Holomorphic Setting

Assume that $X, Y \subset \mathbb{C}^n$ are open connected domains and $\mathbf{F} \subset \mathcal{H}(X)$, $\mathbf{E} \subset \mathcal{H}(Y)$.

Given that $W_{\omega,\Phi} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$, when can we guarantee holomorphicity of ω and Φ?

This question can be decomposed into two parts:

Given that $W_{\omega,\Phi} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$, when can we guarantee continuity of ω and Φ? (Partially answered above)

Given that $W_{\omega,\Phi} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$, and ω and Φ are continuous, when can we guarantee holomorphicity of ω and Φ?

Proposition 3

If $\mathbf{F}$ *is* 2*-independent,* $W_{\omega,\Phi} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$*, and* ω *and* Φ *are continuous, then they are holomorphic.*

Theorem 2 (B.)

If $W_{\omega,\Phi} \in \mathcal{L}(\mathbf{F}, \mathbf{E})$ *and* ω *is non-vanishing, then both* Φ *and* ω *are holomorphic whenever* ***either*** *of the following holds:*

(i) $1 \in \mathbf{F}$ *and* $\mathbf{F}$ *generates the topology of* X*;*

(ii) $\mathbf{F}$ *is* 2*-independent, the bounded functions form a dense set in* $\mathbf{F}$*, and* $\lim_{x \to \partial X} \|\kappa_{\mathbf{F}}(x)\| = +\infty$*.*

References

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THANK YOU!