Elements of an operator theory on the space c_0 over a non-Archimedean valued field
Part I

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Outline for section 1

The Field

Theorem (Hahn, 1907)

Let K be a field (not necessarily ordered) and G a subgroup of $(\mathbb{R},+)$ ordered by the induced order of $\mathbb{R}$. The set

$$K((G)) := \{f : G \to K : supp(f) \text{ is well-ordered}\},$$

where $supp(f) := \{x \in G : f(x) \neq 0\}$, is a field under the addition and multiplication defined as follows: for every $f, g \in K((G))$ and $x \in G$,

1. $(f+g)(x) := f(x) + g(x)$,
2. $fg(x) := \sum_{a+b=x} f(a)g(b)$

*Fields of the form $K((G))$ are called **Hahn fields**.*

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The Valuation

Theorem (Hahn, 1907)

Let G be a subgroup of $(\mathbb{R}, +)$ and let K be any field. If the map $|\ | : K((G)) \to \mathbb{R}$ is defined by

$$|f| := \begin{cases} e^{-\min\{supp(f)\}} & , f \neq 0 \\ 0 & , f = 0, \end{cases}$$

then $(K((G)), |\ |)$ is a Cauchy complete non-Archimedean valued field with value group $|K((G))| = \{e^g \in \mathbb{R} : g \in G\} \cup \{0\}$.

Strong triangle inequality:

$$|x + y| \leq \max\{|x|, |y|\} \text{ for all } x, y \in K.$$

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The Base valued field $\mathcal{H}(i)$

From now on, K will denote a real-closed field i.e.

- $x^2 + 1$ is irreducible in K, and
- $K(i) := K + iK$ is algebraically closed where $i^2 = -1_K$.

Hence $K(i)$ is an algebraic closure of K.

The general Hahn field $K((G))$ will be denoted by $\mathcal{H}$.

Notice that $K(i)((G)) = \mathcal{H} + i\mathcal{H} = \mathcal{H}(i)$. For each nonzero $z = x + iy \in \mathcal{H}(i)$ ($x, y \in \mathcal{H}$) the valuation satisfies:

$$|z| = \max\{|x|, |y|\}$$

The involution $x + iy \mapsto \overline{x + iy} := x - iy$ is an automorphism on $\mathcal{H}(i)$ such that $|z| = |\overline{z}|$ and $z\overline{z} \in \mathcal{H}$ for all $z \in \mathcal{H}(i)$.

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Outline for section 2

The normed space c_0

The set

$$c_0 := c_0(\mathcal{H}(i)) := \left\{ (\lambda_j)_{j\in\mathbb{N}} : \lambda_j \in \mathcal{H}(i), \text{ for all } j \in \mathbb{N}, \lim_j \lambda_j = 0 \right\}$$

is a vector space over $\mathcal{H}(i)$.

Notice that $c_0 = c_0(\mathcal{H}) \oplus i c_0(\mathcal{H})$, i.e. for each $\mathrm{z} = (z_n) \in c_0$, there are unique $\mathrm{x} = (x_n)$ and $\mathrm{y} = (y_n)$ in $c_0(\mathcal{H})$ such that $\mathrm{z} = \mathrm{x} + i\mathrm{y}$ and the norm on c_0 satisfies:

$$||\mathrm{z}|| := \max_{n\in\mathbb{N}} |z_n| = \max_{n\in\mathbb{N}} \max\{|x_n|, |y_n|\} = \max\{||\mathrm{x}||, ||\mathrm{y}||\}.$$

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The space $(c_0, ||\cdot||)$ is Banach.

The inner product on c_0

Proposition

Consider the form $\langle \cdot, \cdot \rangle : c_0 \times c_0 \to \mathcal{H}(i)$, $\langle \mathrm{z}, \mathrm{w} \rangle = \sum_{n=1}^{\infty} z_n \overline{w_n}$. *The statements below hold for all* $\mathrm{z}, \mathrm{z}', \mathrm{w} \in c_0$ *and* $\alpha, \beta \in \mathcal{H}(i)$.

1. $\langle \cdot, \cdot \rangle$ *is well-defined.*
2. $\langle \mathrm{z}, \mathrm{z} \rangle = 0 \Leftrightarrow \mathrm{z} = 0$
3. $\langle \alpha \mathrm{z} + \beta \mathrm{z}', \mathrm{w} \rangle = \alpha \langle \mathrm{z}, \mathrm{w} \rangle + \beta \langle \mathrm{z}', \mathrm{w} \rangle$
4. $\langle \mathrm{z}, \mathrm{w} \rangle = \overline{\langle \mathrm{w}, \mathrm{z} \rangle}$
5. $|\langle \mathrm{z}, \mathrm{w} \rangle| \leq ||\mathrm{z}|| \, ||\mathrm{w}||$
6. $\langle \mathrm{z}, \mathrm{w} \rangle = 0, \forall \mathrm{w} \in c_0 \Rightarrow \mathrm{z} = 0.$
7. $||\mathrm{z}|| = \sqrt{|\langle \mathrm{z}, \mathrm{z} \rangle|}$

Normal and Orthogonal vectors of c_0

Definition

*A subset D of c_0 such that for all $\mathrm{x}, \mathrm{y} \in D$, $\mathrm{x} \neq \mathrm{y} \Rightarrow \langle \mathrm{x}, \mathrm{y} \rangle = 0$, is called a **normal family**. A countable normal family of unit vectors is called an orthonormal sequence. Notice that if $(\mathrm{x}_n)_n$ is an orthonormal sequence in c_0, then $1 = ||\mathrm{x}_n||^2 = |\langle \mathrm{x}_n, \mathrm{x}_\mathrm{n} \rangle|$ but it is not necessary to have $\langle \mathrm{x}_n, \mathrm{x}_\mathrm{n} \rangle = 1$.*

Definition

A subset D of c_0 such that for every finite set $\{\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{k}\} \subset D$,

$$\left\| \sum_{n=1}^{k} \lambda_n \mathrm{x}_n \right\| = \max\{||\lambda_n \mathrm{x}_n|| : 1 \leq n \leq k\},$$

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Normal vs Orthogonal

Proposition (Narici-Beckenstein, 2005)

Every normal family of c_0 is orthogonal. Not every orthogonal family of c_0 is normal.

Definition

*A **base** for c_0 is a orthonormal countable set $(x_n)_n$ in c_0 such that for each $x \in c_0$, there is a unique sequence $(\lambda_n)_n$ in $\mathcal{H}(i)$ satisfying $x = \sum_{n=1}^{\infty} \lambda_n x_n$.*

Definition

*A sequence $(\mathrm{z}_n)_n$ of nonzero vectors of c_0 has the **Riemann-Lebesgue property (RLP)** if for all $\mathrm{w} \in c_0$, $\lim_n \langle \mathrm{z_n}, \mathrm{w} \rangle = 0$.*

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Outline for section 3

Complemented subspaces of c_0

Definition

*Given a subspace M of c_0, the space of all $\mathrm{y} \in c_0$ such that $\langle \mathrm{x}, \mathrm{y} \rangle = 0$ for all $x \in M$ will be denoted by M^p. When $c_0 = M \oplus M^p$, we say that M is **normal complemented** and M^p called the **normal complement** of M.*

Definition

*A continuous linear operator $P \in L(c_0)$ is said to be a **normal projection** if it satisfies the following statements:*

1. $P^2 = P$,
2. $\langle \mathrm{x}, \mathrm{y} \rangle = 0$, *for all* $\mathrm{x} \in \ker(P)$ *and* $\mathrm{y} \in \mathrm{Im}(P)$.

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Complemented subspaces of c_0

Proposition

Let M be an infinite-dimensional closed subspace of c_0. The following statements are equivalent:

1. *M is normal complemented,*
2. *there exists a normal projection $P \in L(c_0)$ such that $\ker(P) = M$.*
3. *every base of M has the RLP,*
4. *M has a base with the RLP.*

Outline for section 4

Characterization of normal projections

Proposition

Let S be any nonempty orthonormal subset of c_0. The operator $P \in L(c_0) \setminus \{0\}$ is a normal projection and $S = \{\mathrm{y}_1, \mathrm{y}_2, \dots, \mathrm{y}_n\}$ is a base for $\mathrm{Im}(P)$ if and only if, for all $\mathrm{x} \in c_0$,

$$P\mathrm{x} = \sum_{i=1}^{n} \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i.$$

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Outline for section 5

Characterization of linear operators with adjoint

Proposition

For each $T \in L(c_0)$ the following statements are equivalent.

1. T *admits an adjoint operator.*
2. $\{T\mathrm{y}_n : n \in \mathbb{N}\}$ *has the RLP for any base* $\{\mathrm{y}_n : n \in \mathbb{N}\}$ *of* c_0*.*
3. *There is a base* $\{\mathrm{y}_n : n \in \mathbb{N}\}$ *of* c_0 *such that* $\{T\mathrm{y}_n : n \in \mathbb{N}\}$ *has the RLP.*
4. $\{T\mathrm{z}_n : n \in \mathbb{N}\}$ *has the RLP for every set* $\{\mathrm{z}_n : n \in \mathbb{N}\} \subset c_0$ *with the RLP.*

Isometries $\ell^\infty \hookrightarrow \mathcal{A}_0$ and $\ell^\infty(\mathcal{H}) \hookrightarrow \mathcal{S}_0$

Definition

Let $\mathcal{A}_0$ be the set of all the operators $T \in L(c_0)$ that admit an adjoint operator T^*, and let $\mathcal{S}_0$ be the set of the self-adjoint operators on c_0.

Proposition

Let $S = \{\mathrm{y}_1, \mathrm{y}_2, \dots\}$ be any base of c_0. The map $\Phi_S : \ell^\infty \to \mathcal{A}_0$, defined by

$$\Phi_S(\lambda) := T \quad \textit{where} \quad T\mathrm{x} := \sum_{i=1}^{\infty} \lambda_i P_i \mathrm{x} \qquad (\lambda = (\lambda_i)_i \in \ell^\infty,\ \mathrm{x} \in c_0)$$

and its restriction to $\ell^\infty(\mathcal{H})$ define isometries $\ell^\infty \hookrightarrow \mathcal{A}_0$ and $\ell^\infty(\mathcal{H}) \hookrightarrow \mathcal{S}_0$, respectively, where for each $i \in \mathbb{N}$, $P_i\mathrm{x} = \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i$, for all $\mathrm{x} \in c_0$.

Isometries $\ell^\infty \hookrightarrow \mathcal{A}_0$ and $\ell^\infty(\mathcal{H}) \hookrightarrow \mathcal{S}_0$

Definition

Let $\mathcal{A}_0$ *be the set of all the operators* $T \in L(c_0)$ *that admit an adjoint operator* T^**, and let* $\mathcal{S}_0$ *be the set of the self-adjoint operators on* c_0*.*

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The spaces $S(\mathcal{A}_0)$ and $S(\mathcal{S}_0)$

Definition

Let $S(\mathcal{A}_0)$ *be the subspace of* $\mathcal{A}_0$ *formed by all the operators* $T \in L(c_0)$ *such that the set of eigenvectors of* T *contains a base of* c_0*.*
Let $S(\mathcal{S}_0)$ *be the subspace of* $S(\mathcal{A}_0)$ *formed by all the self-adjoint operators* $T \in S(\mathcal{A}_0)$*.*

Spectrum and decomposition theorems for $S(\mathcal{A}_0)$

Proposition

Let $T \in S(\mathcal{A}_0)$, *i.e.*

$$T\mathrm{x} := \sum_{i=1}^{\infty} \lambda_i \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i,$$

for each $\mathrm{x} \in c_0$, *where* $(\mathrm{y}_i)_{i\in\mathbb{N}}$ *is a base of* c_0 *and* $\lambda = (\lambda_i)_{i\in\mathbb{N}} \in \ell^\infty$. *Then:*

1. *the set of all the eigenvalues of* T *is* $\{\lambda_n : n \in \mathbb{N}\}$.
2. *the spectrum of* T *is* $\sigma(T) = \overline{\{\lambda_n : n \in \mathbb{N}\}}$.
3. *the spectral radius of* T *is* $||T|| = \sup\{|\lambda_n| : n \in \mathbb{N}\}$.
4. $T \in S(\mathcal{S}_0)$ *if and only if all the eigenvalues of* T *are in* $\mathcal{H}$.
5. *there are unique* $T_1, T_2 \in S(\mathcal{S}_0)$ *such that* $T = T_1 + iT_2$, $T^* = T_1 - iT_2$ *and* $||T|| = \max\{||T_1||, ||T_2||\}$.

Outline for section 6

Compact operators on c_0

Definition

*An operator $T \in L(c_0)$ is called **compact** if there are vectors $\mathrm{a}_1, \mathrm{a}_2, \dots \in c_0$, and functionals $g_1, g_2, \dots \in c_0'$ such that $\lim_i ||g_i|| ||\mathrm{a}_i|| = 0$ and $T = \sum_{i=1}^{\infty} g_i \mathrm{a}_i$, i.e. the sequence $\left(\sum_{i=1}^{n} g_i(\cdot)\mathrm{a}_i\right)_{n\in\mathbb{N}}$ converges uniformly to T.*

Proposition

For any compact operator $T \in L(c_0)$ and any given base $S := \{\mathrm{y}_1, \mathrm{y}_2, \dots\}$ of c_0, there are functionals $f_1, f_2, \dots \in c_0'$ such that $\lim_i ||f_i|| = 0$ and $T = \sum_{i=1}^{\infty} f_i \mathrm{y}_i$.

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Compact operators on c_0

Proposition

Let $T \in L(c_0)$ and let $(\mathrm{y}_i)_{i\in\mathbb{N}}$ be an base of c_0. Then T is compact if and only if $\lim_i \sup_{j\in\mathbb{N}} |\mathrm{y}_i'(T\mathrm{y}_j)| = 0$.

Proposition

Let $T \in L(c_0)$ be an operator with adjoint operator T^*, i.e. $T \in A_0$. If T is compact, so is T^*.

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Proposition

Let $T \in L(c_0)$ *be an operator with adjoint operator* T^**, i.e.* $T \in \mathcal{A}_0$. *If* T *is compact, so is* T^*.

Isometries $c_0 \hookrightarrow \mathcal{A}_1$ and $c_0(\mathcal{H}) \hookrightarrow \mathcal{S}_1$

Definition

Let $\mathcal{A}_1$ be the set of all the compact operators $T \in L(c_0)$ that admit an adjoint, and let $\mathcal{S}_1$ be the set of the compact, self-adjoint operators on c_0.

Proposition

Let $S = \{\mathrm{y}_1, \mathrm{y}_2, \dots\}$ be any base of c_0. The map $\Upsilon_S : c_0 \to \mathcal{A}_1$, defined by

$$\Upsilon_S(\lambda) := T \quad \textit{where} \quad T\mathrm{x} := \sum_{i=1}^{\infty} \lambda_i P_i \mathrm{x} \qquad (\lambda = (\lambda_i)_i \in c_0,\ \mathrm{x} \in c_0)$$

and its restriction to $c_0(\mathcal{H})$ define isometries $c_0 \hookrightarrow \mathcal{A}_1$ and $c_0(\mathcal{H}) \hookrightarrow \mathcal{S}_1$, respectively, where for each $i \in \mathbb{N}$, $P_i \mathrm{x} = \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle} \mathrm{y}_i$, for all $\mathrm{x} \in c_0$.

Isometries $c_0 \hookrightarrow \mathcal{A}_1$ and $c_0(\mathcal{H}) \hookrightarrow \mathcal{S}_1$

Definition

Let $\mathcal{A}_1$ be the set of all the compact operators $T \in L(c_0)$ that admit an adjoint, and let $\mathcal{S}_1$ be the set of the compact, self-adjoint operators on c_0.

Proposition

Let $S = \{\mathrm{y}_1, \mathrm{y}_2, \ldots\}$ be any base of c_0. The map $\Upsilon_S : c_0 \to \mathcal{A}_1$, defined by

$$\Upsilon_S(\lambda) := T \quad \textit{where} \quad T\mathrm{x} := \sum_{i=1}^{\infty} \lambda_i P_i \mathrm{x} \qquad (\lambda = (\lambda_i)_i \in c_0, \ \mathrm{x} \in c_0)$$

and its restriction to $c_0(\mathcal{H})$ define isometries $c_0 \hookrightarrow \mathcal{A}_1$ and $c_0(\mathcal{H}) \hookrightarrow \mathcal{S}_1$, respectively, where for each $i \in \mathbb{N}$, $P_i\mathrm{x} = \frac{\langle \mathrm{x}, \mathrm{y}_i \rangle}{\langle \mathrm{y}_i, \mathrm{y}_i \rangle}\mathrm{y}_i$, for all $\mathrm{x} \in c_0$.

Outline for section 7

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