University of Manitoba
Department of Mathematics
Graduate Comprehensive Exam in Topology

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Friday, 31 January 2020

10:00–16:00 (No extra time allowed)

Instructions

You have six hours to complete the exam.

The exam consists of 3 pages, including this cover page. Answer all eight (8) of the questions in Part A, which is worth a total of 40 marks, distributed according to the table (at right).

For each of Part B and Part C you have a choice of questions. Answer any three (3) of the four (4) 10 mark questions in each part. Parts B and C are worth a total of 30 marks each.

For either Part B or Part C, you may attempt all four questions in that Part; however, only three answers will be evaluated. If you submit responses to all four questions appearing in that part, clearly indicate which responses are to be evaluated. In the absence of any other indication, the first three responses will be evaluated according to the order in which they appear.

To pass this exam, you must obtain a score of at least 75% overall (75 marks out of a possible 100).

Part A:

Question	Marks	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
Total:	40	

Part B:

Question	Marks	Score
9	10	
10	10	
11	10	
12	10	
Total:	30	

Part C:

Question	Marks	Score
13	10	
14	10	
15	10	
16	10	
Total:	30	

Part A: Answer all questions

1. (5 marks) Show that if X is locally path connected and connected, then X is path connected.

2. (5 marks) Show that every second countable topological space is Lindelöf.

3. (5 marks) Define the *one-point compactification* of a locally compact Hausdorff space X. Prove that the one-point compactification of $\mathbb{N}$ (endowed with the discrete topology) is homeomorphic to the subset $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ of $\mathbb{R}$.

4. (5 marks) Let X be a topological space and $D = \{(x,x)|x \in X\} \subset X \times X$ be the diagonal subspace.
   (a) Define the product topology on $X \times X$.
   (b) Show that D is closed if X is Hausdorff.

5. (5 marks) Let $X = \{-1, 0, 1\}^{\mathbb{N}}$, i.e. the set of sequences with values in $\{-1, 0, 1\}$. Show that the function
$$d\left((a_n), (b_n)\right) \;=\; \sup_{n \in \mathbb{N}} |a_n - b_n|$$
makes X into a complete metric space. Show that (X, d) is not sequentially compact.

6. (5 marks) Prove that a compact Hausdorff space is normal.

7. (5 marks) Let $f : X \to Y$ be a continuous, bijective map and Y be Hausdorff. Prove that if X is compact, then f is a homeomorphism.

8. (5 marks) Show that if X is contractible and Y is path connected, then all maps $X \to Y$ are homotopic.

Part B: Point-set topology
Answer any 3 of 4 questions

9. (10 marks) Let (X, d) be a metric space. Fix a subset $Y \subset X$ and define
$$\varphi(x) \;:=\; \inf_{y \in Y} d(x,y) \qquad \forall x \in X.$$
   (a) Show that φ is a continuous function.
   (b) Show that Y is closed iff $Y = \varphi^{-1}(0)$.
   (c) Prove that every metric space is regular.

10. (10 marks) Let X be a topological space.
   (a) Let $A, B \subset X$ be a compact sets. Prove that $A \cup B$ is compact.
   (b) Prove that if X is compact and $A \subset X$ is closed, then A is compact.
   (c) Prove that if X is Hausdorff and $A \subset X$ is compact, then A is closed.

11. (10 marks) Let $X = \mathbb{T}^2$ be the 2-dimensional torus. Let $p, q \in X$ be distinct points and X_1 be the quotient topological space obtained by identifying p and q; let X_2 be the connected sum $X \# X$.
   (a) Is X_1 a topological manifold? Justify your answer with a proof.
   (b) Prove that X_1 and X_2 are not homeomorphic.

12. (10 marks) Let X be a topological space, $A \subset X$ a dense subspace and Y a Hausdorff space. Let $f, g : X \to Y$ be continuous maps such that $f|_A = g|_A$. Prove that $f = g$.

Part C: Algebraic Topology
Answer any 3 of 4 questions

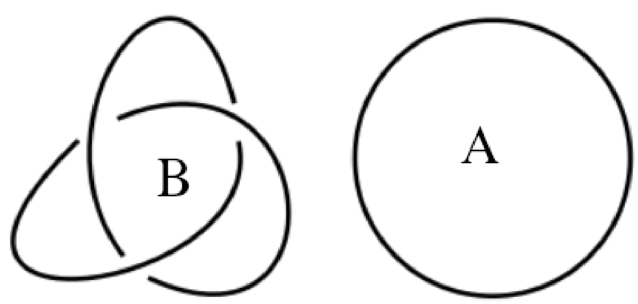
13. (10 marks) In this question, you may use the fact that the fundamental group of a circle is $\mathbb{Z}$ without proof.

    (a) State the Seifert-Van Kampen Theorem.

    (b) For $n \geq 1$, let C_n be a copy of the unit circle and $x_n \in C_n$ a chosen point. Let W denote the quotient of the disjoint union $C_1 \cup C_2$ arising from the identification $x_1 \sim x_2$, we write $W = C_1 \bigvee C_2$ and call W the wedge sum of two circles. Use the Seifert-Van Kampen Theorem to compute $\pi_1(W)$. Conjecture and prove a formula for $\pi_1(\bigvee_{n\in\mathbb{N}} C_n)$, the wedge sum of a countable collection of circles.

    (c) For $n \geq 1$, let D_n denote the circle of radius $1/n$ in $\mathbb{R}^2$ centred at the point $(1/n, 0)$. Let $H = \bigcup_{n=1}^{\infty} D_n$, and give H the subspace topology. Note that all the circles in H intersect at the origin. The group $\pi_1(H)$ is *not* the same as the fundamental group of the wedge of a countable collection of circles. Why does the argument from part (b) not work in this case?

14. (10 marks) Calculate the knot group of each of the knots A and B shown below. Show that the knots are not equivalent.



15. (10 marks) Let $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$ and $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map. Prove that if h is null homotopic, then there exists $y \in \mathbb{S}^1$ such that $h(y) = -y$.

16. (10 marks) Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map such that $\lim\limits_{|x|\to\infty} g(x) = 0$. Prove that g has a fixed point.