Analysis Comprehensive Examination

January 31, 2020

This examination consists of three parts.

- Part A covers the core material. It has 7 questions worth 10 points each, and you must attempt all questions for a total possible score of 70 points.
- Part B covers the specialized material on basic functional analysis. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.
- Part C covers the specialized material on abstract measure and integration. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.

In Part B or C, if you attempt all 3 questions, you must clearly indicate which 2 questions are to be graded. If it is not clearly indicated, the first two questions appearing in the solutions will be graded.

You need to achieve at least 97.5 points (which is 75% of the total 130 possible points on the three parts) in order to pass the examination.

The total time of the examination is six hours. No text or reference books, notes, calculators or aids are allowed in the exam.

PART A

1. Prove that for any real-valued function α increasing on $[0,1]$ we have

$$\lim_{n\to\infty}\int_0^1 x^n\,d\alpha(x) = \alpha(1) - \alpha(1-).$$

2. For $\beta \in \mathbb{R}$, define $f_\beta(x) := \begin{cases} |x|^\beta, & x \neq 0, \\ 0, & x = 0. \end{cases}$

   Find all $\beta \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exist a positive integer n and real numbers $a_0, \ldots, a_n$ satisfying

$$\int_{-\pi}^{\pi}\left|f_\beta(x) - \sum_{k=0}^{n} a_k \cos(kx)\right|^2 dx < \varepsilon.$$

3. For $x \in \mathbb{R}$, consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n^\gamma(1+nx^2)}$.

   (a) If $\gamma > \frac{1}{2}$, show that the series converges uniformly on $\mathbb{R}$.

   (b) If $\gamma > 0$, show that the series converges uniformly on any set $A \subset \mathbb{R}$ such that the closure of A does not contain the origin.

4. (a) State the implicit function theorem.

   (b) Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable function satisfying $F(0,0) = 0$. Use the implicit function theorem to find some condition(s) on F (and/or on its partial derivatives) that guarantees existence of $\varepsilon > 0$ and existence of a continuously differentiable function $g : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$F(F(x, g(x)), g(x)) = 0 \quad \text{for any} \quad x \in (-\varepsilon, \varepsilon).$$

5. Let $a < b$ be real numbers and let $f : [a,b] \to \mathbb{R}$ be a function. Assume that there is a constant M such that for any $x, y \in (a,b)$

$$|f(x) - f(y)| \leq M|x - y|.$$

   Show that f is differentiable almost everywhere on $[a,b]$.

6. For $z = x + iy \in \mathbb{C}\backslash\{0\}$, define

$$u(x,y) = \frac{x}{x^2+y^2}, \quad v(x,y) = -\frac{y}{x^2+y^2}.$$

(a) Show that $f(z) = u(x,y) + iv(x,y)$ is complex analytic in $\mathbb{C}\backslash\{0\}$.

(b) For $a > 0$, $b > 0$, let γ be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

traced once counter-clockwise. Find

$$\int_\gamma f(z)\, dz.$$

7. Let $a_0, \ldots, a_n$ be fixed complex numbers and $p(z) = a_0 + a_1 z + \cdots + a_n z^n$. Let C_R denote the curve $\{z : |z| = R\}$ traced once counter-clockwise. Evaluate

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{C_R} \frac{p'(z)}{p(z)}\, dz.$$

PART B

8. Let $\mathcal{B}$ be the Borel σ-algebra on $[0,1]$.

(a) (5 marks) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that f is $\mathcal{B}$-measurable.

(b) (10 marks) Let $\mu : \mathcal{B} \to [0,\infty]$ be a measure. Let $g, h : [0,1] \to \mathbb{R}$ be continuous functions. Assume that $g(t) = h(t)$ for $[\mu]$-almost every $t \in [0,1]$. Show that $g = h$.

9. Let $(X, \mathfrak{T}, \mu)$ be a measure space, that is, X is a set, $\mathfrak{T}$ is a σ-algebra on X, and μ is a measure on $(X, \mathfrak{T})$. For each positive integer $n \geq 1$, let $f_n : X \to \mathbb{R}$ be a $\mathfrak{T}$-measurable function. Assume that there is a μ-integrable function $b : X \to \mathbb{R}$ with the property that $f_n(x) \leq b(x)$ for $[\mu]$-almost every $x \in X$. Show that

$$\limsup_{n\to\infty} \left(\int_X f_n d\mu \right) \leq \int_X \left(\limsup_{n\to\infty} f_n \right) d\mu.$$

10. Let $(X, \mathfrak{T}, \mu)$ be a measure space, that is, X is a set, $\mathfrak{T}$ is a σ-algebra on X, and μ is a measure on $(X, \mathfrak{T})$. Assume that μ is a finite measure.

(a) (5 marks) Let $\phi : X \to \mathbb{R}$ be a μ-integrable function. Define $\sigma_\phi : \mathfrak{T} \to \mathbb{R}$ as

$$\sigma_\phi(E) = \int_E \phi \, d\mu, \quad E \in \mathfrak{T}.$$

Assume that σ_ϕ is a measure which is singular with respect to μ. Find the value of $\sigma_\phi(X)$.

(b) (10 marks) For each positive integer $n \geq 1$, let $f_n : X \to [0,\infty)$ be a bounded $\mathfrak{T}$-measurable function. Assume that the sequence of functions (f_n) converges to 0 pointwise $[\mu]$-almost everywhere on X. Show that

$$\lim_{n\to\infty} \int_X f_n d\nu = 0$$

whenever ν is a finite measure which is absolutely continuous with respect to μ.

PART C

11. Let $n \geq 1$ be a positive integer and let X be a normed space with dimension n. Let X^* denote the space of continuous linear functionals on X.

    (a) (5 marks) Show that X^* has dimension at most n.

    (b) (10 marks) Show that there is an isometric linear isomorphism between X and X^{**}. [*Hint*: use (a) along with the canonical embedding of X into X^{**}.]

12. Let X be a Banach space and let $P : X \to X$ be a linear operator such that $P^2 = P$.

    (a) (5 marks) Show that $PX = \ker(I - P)$.

    (b) (10 marks) Show that P is bounded if and only if $\ker P$ and PX are closed. [*Hint*: use the Closed Graph Theorem along with part (a).]

13. Let X be a normed space and let Y be a Banach space. Let $X_0 \subset X$ be a dense subspace. Let $T : X_0 \to Y$ be a continuous linear operator. Show that there is a continuous linear operator $\widehat{T} : X \to Y$ with the property that $\widehat{T}|_{X_0} = T$.