University of Manitoba  
Department of Mathematics

Graduate Comprehensive Examination in Topology

September 19th, 2019 (10 am - 4 pm)

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Instructions (Please read carefully):

• You have altogether six hours to complete the examination. The question paper consists of seven pages, including this cover page.

• You should answer all the eight questions in Part A. They are worth 40 marks in total, distributed according to the following table:

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• In Parts B and C, you have a choice of questions. In each of these parts, you should answer any three of the four questions (worth 10 marks each). You may attempt all four of them; however you should indicate clearly which three should be marked. If no such indication is given, then we will mark the first three solutions in the order they appear.

• Part A is worth 40 marks and parts B,C are worth 30 marks each, making a total of 100 marks for the entire examination. You must obtain a score of at least 75% in order to pass.

I have submitted ______ numbered pages together with this question paper.

Student’s Signature:

Invigilator’s Signature:
PART A

Please answer each of the following eight questions. You should give adequate justifications for your answers.

Q1. Let $X$ be a topological space. Let $A$ and $B$ denote subspaces of $X$ such that $X = A \cup B$. Let $U \subseteq A \cap B$ be a subset such that $U$ is open in $A$ as well as in $B$. Prove that $U$ is open in $X$. [4]

Q2. Determine whether the topological spaces $X$ and $Y$ shown in the diagram below are homeomorphic. You must give an adequate justification for your answer. [5]

![Diagram of topological spaces X and Y]

Q3. Show that the following two conditions are equivalent for a topological space $X$. [4]

1. Every countable open cover of $X$ has a finite subcover.

2. If $\{Y_i\}_{i=1}^{\infty}$ is a collection of closed subsets of $X$ with $\bigcap_{i=1}^{\infty} Y_i = \emptyset$, then there exists a finite subset $F \subseteq \{1, 2, 3, \ldots\}$ such that $\bigcap_{i \in F} Y_i = \emptyset$.

Q4. Let $N = \{0, 1, 2, \ldots\}$ denote the set of natural numbers with discrete topology, and let $N^*$ denote the one-point compactification of $N$. Show that $N^*$ is homeomorphic to a subspace of $\mathbb{R}$. [5]
Q5. State the definitions of the regularity conditions $T_0$, $T_1$ and $T_2$. [4]

1. Give an example of a topological space which is $T_0$ but not $T_1$.
2. Give an example of a topological space which is $T_1$ but not $T_2$.

Q6. [6]

1. Define what it means to say that a topological space is first countable.
2. Give an example of a topological space which is not first countable.

Q7. [6]

1. Define what it means to say that a topological space is regular.
2. Show that if $(X, d)$ is a metric space, then $X$ is regular.

Q8. This question is on the subject of compact 2-manifolds without boundary. Let $T = S^1 \times S^1$ and $\mathbb{RP}^2$ denote the torus and the real projective plane respectively. Given two manifolds $X$ and $Y$, let $X \# Y$ denote their connected sum. Now, for each of the manifolds $M$ listed below, state the fundamental group of $M$ and state whether $M$ is orientable or non-orientable. A bare answer is sufficient; no explanation is necessary. [6]

\[ M = T \# \mathbb{RP}^2 \quad \text{and} \quad M = T \# T. \]
PART B (point-set topology)

You should answer any three of the following four questions in your answer booklet. Each question is worth 10 marks. If you attempt more than three, then please indicate clearly which ones you want us to mark. In the absence of any such indication, we will mark the first three.

Q9. Let \( f : X \to Y \) be a continuous map of topological spaces. Define a subspace \( \Gamma \subseteq X \times Y \) as follows:

\[
\Gamma = \{(x, y) \in X \times Y : f(x) = y\}.
\]

1. Show that \( \Gamma \) is homeomorphic to \( X \).
2. Assuming that \( Y \) is Hausdorff, prove that \( \Gamma \) is closed in \( X \times Y \).

Q10. Define what it means to say that a topological space \( X \) is separable.

1. Prove that a compact metric space is separable.

Q11. Suppose that \( X \) is a normal topological space with a dense subset \( Y \), and that \( D \subseteq X \) is a closed, discrete subspace. Show that

\[
|D| < |\mathbb{R}^Y|.
\]

Hint: Use Tietze’s extension theorem to count the continuous functions \( X \to \mathbb{R} \) which arise from extending continuous functions \( D \to \mathbb{R} \).
Q12. Define what it means to say that a topological space is connected. [10]

Now let $T_1$ and $T_2$ be topologies on a set $X$ such that $T_1 \subseteq T_2$ (i.e., $T_1$ is weaker than $T_2$).

(a) Is $(X, T_1)$ necessarily connected if $(X, T_2)$ is connected?

(b) Is $(X, T_2)$ necessarily connected if $(X, T_1)$ is connected?

In (a) or (b), if your answer is ‘yes’, then give a proof. If it is ‘no’, then give a counterexample.
PART C (algebraic topology)

You should answer any three of the following four questions in your answer booklet. Each question is worth 10 marks. If you attempt more than three, then please indicate clearly which ones you want us to mark. In the absence of any such indication, we will mark the first three.

Q13. The following two diagrams show the Hopf link and the Whitehead link respectively. Calculate the link group of each, and thereby show that these two links are not equivalent.  [10]

Q14. Consider the space $X = \mathbb{RP}^2 \times S^1$, where $\mathbb{RP}^2$ denotes the real projective plane and $S^1$ is the circle in its usual topology. [10]

1. State (without proof) the fundamental group of $X$.

2. Describe all the covering maps of $X$. In other words, give a list of covering maps

$$f_i : Y_i \to X,$$

such that any other covering map $g : Z \to X$ is equivalent to exactly one of the $f_i$.

You should give a reasonably adequate justification as to why your list is complete, but you may assume standard facts about covering spaces.
Q15.  

1. State the Seifert van-Kampen theorem.

2. Let $A$ denote the topological space $[0, 1] \times [0, 1]$ with its usual topology inherited from $\mathbb{R}^2$. Define an equivalence relation $\sim$ on $A$ as follows. We have $(x_1, y_1) \sim (x_2, y_2)$ if and only if, either of the following conditions is true:
   
   - $x_1 = x_2$ and $\{y_1, y_2\} = \{0, 1\}$,
   - $\{x_1, x_2\} = \{0, 1\}$ and $y_1 + y_2 = 1$.

Let $B = A/\sim$ denote the space with the resulting quotient topology. Now illustrate the use of the Seifert van-Kampen theorem by calculating the fundamental group of $B$.

Q16. Let $X$ be a topological space, and let $A \subseteq X$ be a subspace.  

1. Define the notion of a deformation retraction of $X$ onto $A$.

2. Now consider the spaces
   
   $$X = \mathbb{R}^2 \setminus \{(1/2, 1/2)\}, \quad \text{and} \quad A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$  

   Define an explicit deformation retraction of $X$ onto $A$.

3. Let
   
   $$Y = \mathbb{R}^2 \setminus \{(1/2, 1/2), (-1/2, -1/2)\}.$$  

   Give a concise proof of the fact that there is no deformation retraction of $Y$ onto $A$.  
