# University of Manitoba
# Department of Mathematics

## Graduate Comprehensive Examination in Topology

September 19th, 2019 (10 am - 4 pm)

**Examiners:** J. Chipalkatti (co-ordinator), A. Clay, S. Kalajdzievski.

**Instructions (Please read carefully):**

- You have altogether six hours to complete the examination. The question paper consists of seven pages, including this cover page.
- You should answer all the eight questions in Part A. They are worth $40$ marks in total, distributed according to the following table:

| Question | Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 | Q8 |
|---|---|---|---|---|---|---|---|---|
| Marks | 4 | 5 | 4 | 5 | 4 | 6 | 6 | 6 |

- In Parts B and C, you have a choice of questions. In each of these parts, you should answer any three of the four questions (worth $10$ marks each). You may attempt all four of them; however you should indicate clearly which three should be marked. If no such indication is given, then we will mark the first three solutions in the order they appear.
- Part A is worth $40$ marks and parts B,C are worth $30$ marks each, making a total of $100$ marks for the entire examination. You must obtain a score of at least $75\%$ in order to pass.

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I have submitted _______ numbered pages together with this question paper.

Student's Signature:

Invigilator's Signature:

# PART A

Please answer each of the following eight questions. You should give adequate justifications for your answers.

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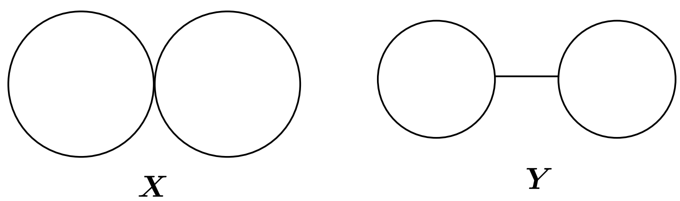
Q1. Let $X$ be a topological space. Let $A$ and $B$ denote subspaces of $X$ such that $X = A \cup B$. Let $U \subseteq A \cap B$ be a subset such that $U$ is open in $A$ as well as in $B$. Prove that $U$ is open in $X$. [4]

Q2. Determine whether the topological spaces $X$ and $Y$ shown in the diagram below are homeomorphic. You must give an adequate justification for your answer. [5]




Q3. Show that the following two conditions are equivalent for a topological space $X$. [4]

1. Every countable open cover of $X$ has a finite subcover.
2. If $\{Y_i\}_{i=1}^{\infty}$ is a collection of closed subsets of $X$ with $\bigcap_{i=1}^{\infty} Y_i = \emptyset$, then there exists a finite subset $F \subseteq \{1, 2, 3, \dots\}$ such that $\bigcap_{i \in F} Y_i = \emptyset$.

Q4. Let [5]

$$N = \{0, 1, 2, \dots\}$$

denote the set of natural numbers with discrete topology, and let $N^*$ denote the one-point compactification of $N$. Show that $N^*$ is homeomorphic to a subspace of $\mathbb{R}$.

Q5. State the definitions of the regularity conditions $T_0, T_1$ and $T_2$. [4]

1. Give an example of a topological space which is $T_0$ but not $T_1$.
2. Give an example of a topological space which is $T_1$ but not $T_2$.

Q6. [6]

1. Define what it means to say that a topological space is first countable.
2. Give an example of a topological space which is not first countable.

Q7. [6]

1. Define what it means to say that a topological space is regular.
2. Show that if $(X, d)$ is a metric space, then $X$ is regular.

Q8. This question is on the subject of compact 2-manifolds without boundary. Let $\mathbb{T} = S^1 \times S^1$ and $\mathbb{RP}^2$ denote the torus and the real projective plane respectively. Given two manifolds $X$ and $Y$, let $X \# Y$ denote their connected sum. Now, for each of the manifolds $M$ listed below, state the fundamental group of $M$ and state whether $M$ is orientable or non-orientable. A bare answer is sufficient; no explanation is necessary. [6]

$$M = \mathbb{T} \# \mathbb{RP}^2 \qquad \text{and} \qquad M = \mathbb{T} \# \mathbb{T}.$$

## PART B (point-set topology)

You should answer any three of the following four questions in your answer booklet. Each question is worth $10$ marks. If you attempt more than three, then please indicate clearly which ones you want us to mark. In the absence of any such indication, we will mark the first three.

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Q9. Let $f : X \longrightarrow Y$ be a continuous map of topological spaces. Define a subspace $\Gamma \subseteq X \times Y$ as follows: [10]

$$\Gamma = \{(x, y) \in X \times Y : f(x) = y\}.$$

1. Show that $\Gamma$ is homeomorphic to $X$.
2. Assuming that $Y$ is Hausdorff, prove that $\Gamma$ is closed in $X \times Y$.

Q10. [10]

1. Define what it means to say that a topological space $X$ is separable.
2. Prove that a compact metric space is separable.

Q11. Suppose that $X$ is a normal topological space with a dense subset $Y$, and that $D \subseteq X$ is a closed, discrete subspace. Show that [10]

$$|D| < |\mathbb{R}^Y|.$$

Hint: Use Tietze's extension theorem to count the continuous functions $X \longrightarrow \mathbb{R}$ which arise from extending continuous functions $D \longrightarrow \mathbb{R}$.

Q12. Define what it means to say that a topological space is connected. [10]

Now let $T_1$ and $T_2$ be topologies on a set $X$ such that $T_1 \subseteq T_2$ (i.e., $T_1$ is weaker than $T_2$).

(a) Is $(X, T_1)$ necessarily connected if $(X, T_2)$ is connected?

(b) Is $(X, T_2)$ necessarily connected if $(X, T_1)$ is connected?

In (a) or (b), if your answer is 'yes', then give a proof. If it is 'no', then give a counterexample.

## PART C (algebraic topology)

You should answer any three of the following four questions in your answer booklet. Each question is worth $10$ marks. If you attempt more than three, then please indicate clearly which ones you want us to mark. In the absence of any such indication, we will mark the first three.

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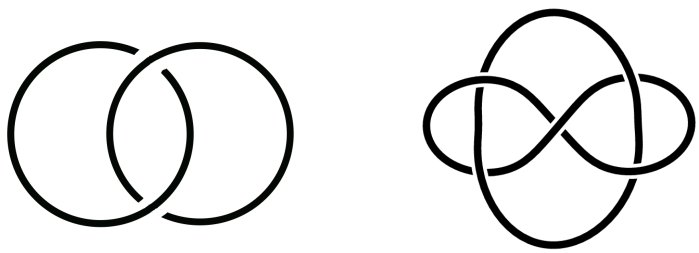
Q13. The following two diagrams show the Hopf link and the Whitehead link respectively. Calculate the link group of each, and thereby show that these two links are not equivalent. [10]



Q14. Consider the space $X = \mathbb{RP}^2 \times S^1$, where $\mathbb{RP}^2$ denotes the real projective plane and $S^1$ is the circle in its usual topology. [10]

1. State (without proof) the fundamental group of $X$.
2. Describe all the covering maps of $X$. In other words, give a list of covering maps

$$f_i : Y_i \longrightarrow X,$$

such that any other covering map $g : Z \longrightarrow X$ is equivalent to exactly one of the $f_i$.

You should give a reasonably adequate justification as to why your list is complete, but you may assume standard facts about covering spaces.

Q15. [10]

1. State the Seifert van-Kampen theorem.

2. Let $A$ denote the topological space $[0,1] \times [0,1]$ with its usual topology inherited from $\mathbb{R}^2$. Define an equivalence relation $\sim$ on $A$ as follows. We have $(x_1, y_1) \sim (x_2, y_2)$ if and only if, either of the following conditions is true:

   - $x_1 = x_2$ and $\{y_1, y_2\} = \{0, 1\}$,
   - $\{x_1, x_2\} = \{0, 1\}$ and $y_1 + y_2 = 1$.

   Let $B = A/\sim$ denote the space with the resulting quotient topology. Now illustrate the use of the Seifert van-Kampen theorem by calculating the fundamental group of $B$.

Q16. Let $X$ be a topological space, and let $A \subseteq X$ be a subspace. [10]

1. Define the notion of a deformation retraction of $X$ onto $A$.

2. Now consider the spaces

$$X = \mathbb{R}^2 \setminus \{(1/2, 1/2)\}, \qquad \text{and} \qquad A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

   Define an explicit deformation retraction of $X$ onto $A$.

3. Let

$$Y = \mathbb{R}^2 \setminus \{(1/2, 1/2), (-1/2, -1/2)\}.$$

   Give a concise proof of the fact that there is no deformation retraction of $Y$ onto $A$.

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