UNIVERSITY OF MANITOBA

COMPREHENSIVE EXAMINATION

DATE: October 4, 2019 TIME: 6 hours

EXAMINATION: DE EXAMINER: L. Butler, S. Lui and S. Portet (coordinator)

Question	Points	Score
1	8	
2	7	
3	13	
4	10	
5	5	
6	7	
7	8	
8	15	
9	6	
10	7	
11	4	
12	10	
Total:	100	

INSTRUCTIONS TO STUDENTS:

This is a 6 hour examination. **No extra time will be given.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

The total value of all questions is 100 points, with 50 marks on each of Ordinary and Partial Differential Equations (ODE and PDE) parts. The passing mark is 75 marks (75% of the total 100 points).

The ODE and PDE parts have the following structure, for a subtotal of 50 marks:

1. The ODE part is composed of 6 questions, questions 1-6. The detail of marks is given in the table above.
2. The PDE part is composed of 6 questions, questions 7-12. The detail of marks is given in the table above.

The value of each question is indicated in the left margin beside the statement of the question.

Please detail carefully your work.

Ordinary differential equations

1. Let

$$\frac{dx}{dt} = f(x,t), \qquad x(t_0) = x_0. \qquad (IVP)$$

[3] (a) State the existence and uniqueness theorem for (IVP).

[5] (b) Let $f(x,t) = x^{2/3}$ for $(x,t) \in \mathbb{R}^2$ and $(x_0, t_0) = (0,0)$. Prove that there exist infinitely many solutions to (IVP).

2. Let

$$\frac{dx}{dt} = x^2 - a, \qquad x(0) = x_0,$$

for $x(t), a \in \mathbb{R}$.

[4] (a) Sketch the bifurcation diagram.

[3] (b) Based on the bifurcation diagram or otherwise, determine the omega limit set of the point $x_0 \in \mathbb{R}$.

3. Consider the planar systems of ODE

$$\frac{dx}{dt} = y + ax(1 - x^2 - y^2),$$
$$\frac{dy}{dt} = -x + ay(1 - x^2 - y^2).$$

[5] (a) Determine all periodic orbits.

[4] (b) Use the linearization method to determine the stability of each fixed point as a function of a.

[4] (c) Use Lyapunov's method with the function $V = x^2 + y^2$ to determine the stability of each non-trivial periodic orbit.

4. Let

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \qquad (LDE)$$

[3] (a) Determine the set of fixed points of (LDE) and the stability of each fixed point.

[2] (b) Determine the alpha limit set of the point (x_0, y_0, z_0).

[2] (c) Show that the x-y plane and z axis are invariant sets.

[3] (d) Sketch the phase portrait of (LDE).

[5] 5. Solve the initial-value problem

$$\frac{dy}{dt} + 2\frac{y}{t} = t^2 y^2, \qquad y(1) = 1.$$

6. Let $\epsilon > 0$ and

$$y'' + 2\epsilon y' + y = \sin(wt) \qquad (DE).$$

[5] (a) Use the Laplace transform to solve the initial-value problem with $y(0) = y'(0) = 0$.

[2] (b) Prove that if $y = y(t)$ is a solution to (DE) and $y_p = y_p(t)$ is the solution found in the first part, then $|y(t) - y_p(t)| \to 0$ as $t \to \infty$.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \|a\|$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \|a\|$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}

Partial Differential Equations

[8] 7. Solve the PDE $u_t + 3u_x = 0$ in the quarter plane $x, t > 0$ with initial condition $u(x,0) = u_0(x),\ x > 0$ and boundary condition $u(0,t) = f(t),\ t > 0$. Assume u_0 and f are continuously differentiable on $[0,\infty)$. Find necessary and sufficient condition(s) on u_0 and f so that the solution is continuously differentiable in the quarter plane.

[15] 8. Solve the PDE $u_t = u_{xx}$ on the strip $x \in (0,1),\ t > 0$ with boundary conditions $u_x(0,t) = 0,\ u(1,t) = 4,\ t > 0$ and initial condition $u(x,0) = 4x^2,\ x \in (0,1)$. Solve this problem by writing $u(x,t) = z(x) + v(x,t)$, where v satisfies homogeneous boundary conditions. Find z and then solve the PDE for v by first solving the associated homogeneous PDE.

[6] 9. Consider the PDE $u_t + au = ku_{xx} + f,\ x \in \mathbb{R},\ t > 0$ with initial condition $u(x,0) = u_0(x)$. Assume that there is a unique smooth solution u, and a, u_0, f are even smooth functions of x and k is a positive constant. Show that $u(x,t)$ is an even function of x for all $t > 0$.

[7] 10. Consider the PDE $u_{tt} = u_{xx} + \alpha u_{xxt}$ for $x \in (0,1)$ with initial conditions $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$ and boundary conditions $u(0,t) = u(1,t) = 0$ for $t > 0$. Here α is a real constant and u is smooth. Define the energy

$$E(t) = \frac{1}{2}\int_0^1 u_t^2(x,t)dx + \frac{1}{2}\int_0^1 u_x^2(x,t)dx.$$

Find a condition on α so that $E'(t) \le 0$ for all $t \ge 0$. (Full marks will not be given if too strong a condition, such as $\alpha = 0$, is given.) Assuming that this condition holds, show that the PDE has at most one classical solution.

[4] 11. Let Ω be a bounded domain in $\mathbb{R}^2$ with a smooth boundary. Define

$$\|u\|_2^* = \left(\int_\Omega (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2)\, dxdy\right)^{1/2}, \qquad u \in C_0^\infty(\Omega),$$

where $C_0^\infty(\Omega)$ is the space of $C^\infty(\Omega)$ functions compactly supported in Ω. Show that

$$\|\Delta u\|_{L^2} = \|u\|_2^*, \qquad u \in C_0^\infty(\Omega).$$

[10] 12. Let Ω be a bounded domain on $\mathbb{R}^N$ with a smooth boundary. Assume $N \ge 3$. Consider the PDE $-\Delta u + cu = f$ on Ω for a smooth solution u vanishing on the boundary $\partial\Omega$. Assume f is smooth and non-negative on Ω and c is also smooth. Let $c_-(x) = \max(0, -c(x))$ be the negative part of c. Assume there is some positive constant M so that $c_- \le M$ on Ω. Find a positive ϵ so that if $|\Omega| < \epsilon$, then $u \ge 0$ on Ω. Hint: Use the inequality

$$\|v\|_{L^s} \le C\,\|\nabla v\|_{L^2}, \qquad v \in C_0^\infty(\Omega),$$

where C is a constant and $s = 2N/(N-2)$.