University of Manitoba
Department of Mathematics

Graduate Comprehensive Examination in Algebra

10 AM–4 PM 30 September, 2019.

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Instructions (Please read carefully):

- You have altogether 6 hours to complete the examination.
- Part A consists of 10 questions worth two marks each. Answer all questions in Part A on the question paper itself. Each of these questions can and should be answered in no more than three sentences.
- You have a choice of questions in each of Parts B and C. The questions in Part B are worth 5 marks each. Answer any 6 questions out of 10 in this part. The questions in Part C are worth 10 marks each. Answer any 4 questions out of 6 in this part.
- You may attempt as many questions as you like in Parts B and C; however, if you attempt more than the required number of questions, you must clearly indicate which answers you want evaluated. In the absence of any explicit indication, the first 6 questions for Part B, and the first 4 questions for Part C (in the order of their appearance in your answer booklets) will be evaluated.
- In order to pass this examination, you must obtain a score of at least 75% in total.

Be sure to keep in mind the following:

- Unless stated otherwise, vector spaces need not be finite dimensional.
- Unless stated otherwise, groups may be finite or infinite, abelian or non-abelian.
- Unless stated otherwise, rings may be commutative or non-commutative.
- Unless stated otherwise, rings R **are** assumed to have a multiplicative identity $1 \in R$.
- Unless stated otherwise, fields may be finite or infinite, of arbitrary characterstic.
- S_n denotes the group of permutations on the set $\{1, \ldots, n\}$.

PART A

Please answer each of the following 10 questions in the space provided. Each correct answer is worth two marks. Each question should be answered briefly; i.e., in no more than three sentences.

A1. Prove or disprove: For any prime number p, the group $\mathbb{Z}_p \times \mathbb{Z}_p$ is a cyclic group.

A2. Prove or disprove: If a ring R is a principal ideal domain then the polynomial ring $R[x]$ is also a principal ideal domain.

A3. List all elements of the group $\mathbb{Z}_{14}^{\times}$. Is the group $Z_{14}^{\times}$ cyclic?

A4. Let K be a field. Define what it means for an extension $\overline{K}$ of K to be *an algebraic closure* of K.

A5. Show that the only ideals of a field F are (0) and F itself.

A6. Define what it means for an extension of fields $F \subset K$ to be *normal*.

A7. Give an example (with a brief explanation) of a finite field extension that is *not* Galois.

A8. What are the possible Jordan canonical forms for a matrix A with characteristic polynomial $p_A(t) = (t-3)^3(t+5)$?

A9. State a composition series of $S_3 \times \mathbb{Z}/2\mathbb{Z}$, and the associated quotients.

A10. Suppose that V is a finite dimensional vector space. Write down an isomorphism $T : V \to V^*$, where V^* is the dual of V. Note that you need only describe the map T, you do not need to prove it is an isomorphism.

PART B

Please answer any 6 of the following 10 questions in your answer booklet. Each question is worth 5 marks. If you attempt more than 6 questions, then please indicate clearly which ones you want evaluated.

B1. Let G be a group. Show that if $|G| = 616$ then G is not simple.

B2. Let R be a ring such that $a^2 = -a$ for all $a \in R$. Prove that R is abelian.

B3. For a fixed element a in a group G let $H_a = \{x \in G \mid xa = ax\}$.

1. Prove that H_a is a subgroup of G.
2. If $G = S_5$ and $a = (1\,2\,3)$, find two nontrivial elements of H_a beside a.

B4. Let D be a Euclidean domain with norm δ, and assume that $\delta(x)\delta(y) = \delta(xy)$ for all $x, y \in D$. First show that $\delta(1) = 1$, and then prove that if $\delta(a)$ is a prime number, then a is irreducible in D.

B5. Show that $\mathbb{Q}(\sqrt[7]{10})$ has no proper subfields besides $\mathbb{Q}$.

B6. Give an example of a ring R and a projective R-module that is not free.

B7. Suppose R is an integral domain, and M an R-module. A element $m \in M$ is called *torsion* if there exists $r \in R$ such that $r \cdot m = 0$. A module is called *torsion-free* if there are no non-zero torsion elements. Show that the set M_{tors} of torsion elements is a submodule of M, and that M/M_{tors} is torsion-free.

B8. Consider the symmetric group S_3.

1. Prove that $\mathrm{Inn}(S_3) \cong S_3$.
2. Determine $\mathrm{Aut}(S_3)$.

B9. State the Cayley-Hamilton theorem. Prove that it holds for matrices that are in Jordan canonical form.

B10. Let $F \subset K$ be an extension of fields such that $[K : F] = 2$ and suppose that $char(F) \neq 2$. Show that there exists $a \in K$ such that $K = F(a)$ and $a^2 \in F$.

PART C

Please answer any 4 of the following 6 questions in your answer booklet. Each question is worth 10 marks. If you attempt more than 4 questions, then please indicate clearly which ones you want evaluated.

C1. Let G be a group.

1. Prove that if $\dfrac{G}{Z(G)}$ is cyclic then G must be abelian.
2. Prove that if G is a non abelian group of order p^3, where p is a prime number, then $|Z(G)| = p$.

C2. First define a principal ideal domain and a unique factorization domain, and then prove that any principal ideal domain is a unique factorization domain (Hint: You may use without proof the fact that principal ideal domains are Noetherian).

C3. Prove, using Zorn's lemma, that every vector space has a basis.

C4. Let $f(x) = x^3 - 2 \in \mathbb{Q}[x]$.

1. Show that $f(x)$ is irreducible.
2. Give an explicit presentation (with generators and relations) for the Galois group $Gal(K/\mathbb{Q})$, where K is the splitting field of f.

C5. Suppose R is a commutative ring, and M is an R-module.

1. Define what it means for M to be *flat*.
2. Suppose $f : R \to S$ is a flat morphism of commutative rings (i.e. S is flat as an R-module), and $I \subset R$ is an ideal. Show that there is an isomorphism $I \otimes_R S \simeq IS$.
3. Find an example of rings R and S, a morphism $f : R \to S$, and an ideal $I \subset R$ where conclusion of part (2) fails.

C6. Suppose that R and S are commutative Artinian rings. Show that $R \times S$ is also an Artinian ring.

—END OF EXAM—