Analysis Comprehensive Examination

Department of Mathematics, University of Manitoba

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This examination consists of three parts.

- Part A covers the core material. It has 8 questions worth 10 points each. You must attempt all questions in this part for a total possible score of 80 points.
- Part B covers the specialized material on abstract measure and integration. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.
- Part C covers the specialized material on basic functional analysis. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.

If you attempt more than the required number of questions in Part B or C, you must clearly indicate which questions are to be graded. If it is not clearly indicated, solutions to those appearing first in the booklet will be graded.

You need to achieve at least 105 points, which is 75% of the total possible 140 points, in order to pass the examination.

The total time of the examination is six hours. No books, notes, calculators or aids are allowed during the exam.

Part A

Solve all the problems in this part.

1. Let $n \geq 1$ be an integer and let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a function with continuous partial derivatives. Let $a \in \mathbb{R}^n$. Assume that 0 is not an eigenvalue of $\mathbf{F}'(a)$. Show that there is an open neighbourhood $U \subset \mathbb{R}^n$ of a such that
$$U \cap \{x \in \mathbb{R}^n : \mathbf{F}(x) = \mathbf{F}(a)\} = \{a\}.$$

2. Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal set of functions in $L^2[0,1]$. Let $\mathcal{M} \subset L^2[0,1]$ be the subspace generated by $\varphi_1, \varphi_2, \varphi_3, \varphi_4$. Let $f \in L^2[0,1]$ and let c_1, c_2, c_3, c_4 be real numbers such that
$$\left\| f - \sum_{k=1}^{4} c_k \varphi_k \right\| = \inf_{g \in \mathcal{M}} \|f - g\|.$$
Calculate
$$\langle f, \varphi_1 + 2\varphi_2 + 3\varphi_3 + 4\varphi_4 \rangle.$$

3. Let α be an increasing function and f be a real-valued function on $[a,b]$. If f is Riemann-Stieltjes integrable on $[a,b]$ with respect to α, show that the function $G(x) = \int_a^x f d\alpha$ is of bounded variation on $[a,b]$.

4. Show that the series $\sum_{n=0}^{\infty} e^{-nx} \cos(nx)$ converges for every $x > 0$. If $f(x)$ is the sum function of the series, prove
$$f'(x) = -\sum_{n=0}^{\infty} ne^{-nx}(\cos(nx) + \sin(nx))$$
for every $x > 0$. (You may use a theorem on term-by-term differentiation for a function series. If you use such a theorem, include a full statement of it in your solution.)

5. (a) State the Weierstrass approximation theorem.

   (b) Show that if the function $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and if
$$\int_a^b x^n f(x)\, dx = 0,$$
   for each $n = 0, 1, 2, ...$, then f is identically 0 on $[a,b]$.

6. Find the Laurent expansion of the function $f(z) = \frac{1}{z(z-2)^2}$ around $z_0 = 0$ on the following regions.

   (a) $A = \{z : 0 < |z| < 2\}$

(b) $B = \{z : 2 < |z|\}$.

7. (a) State Rouché's Theorem.

   (b) Use (a) to show that if f is analytic on the closed unit disk, and $0 < |f(z)| < 1$ on the unit circle, then f has exactly one fixed point on the unit disk.

   (Hint: consider the functions $g(z) = f(z) - z$ and $h(z) = -z$.)

8. (a) State the Residue Theorem.

   (b) Let γ be the positively oriented circle centered at the origin and with radius 4. Evaluate the integral
$$I = \int_\gamma \frac{e^{2z}}{\sin z} dz \,.$$

Part B

Solve 2 of the following 3 problems.

1. Let $(X, \mathcal{A}, \mu_i)$ and $(Y, \mathcal{B}, \nu_i)$, $i = 1, 2$, be complete σ-finite measure spaces. Suppose that μ_1 is absolutely continuous with respect to μ_2 and ν_1 is absolutely continuous with respect to ν_2. Show that $\mu_1 \times \nu_1$ is absolutely continuous with respect to $\mu_2 \times \nu_2$. (Hint: use the Radon-Nikodym Theorem.)

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and f be a μ-integrable function on X. Show that $E = \{x \in X : f(x) \neq 0\}$ is a σ-finite $\mathcal{M}$-measurable set.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume that μ is *semifinite*, i.e. for each $E \in \mathcal{M}$ such that $\mu(E) = \infty$, there is $F \in \mathcal{M}$ such that $F \subset E$ and $0 < \mu(F) < \infty$. Let $E \in \mathcal{M}$ and $\mu(E) = \infty$. Show that, for each $k > 0$, there exists $F \in \mathcal{M}$ such that $F \subset E$ and $k < \mu(F) < \infty$. (Hint: Consider $\mathcal{F} = \{F \in \mathcal{M} : F \subset E, 0 < \mu(F) < \infty\}$ and $\sup\{\mu(F) : F \in \mathcal{F}\}$.)

Part C

Solve 2 of the following 3 problems.

1. Let X be a normed space and let $M \subset X$ be a closed subspace. Let $q : X \to X/M$ denote the quotient map. Find the norm of q as a linear operator. Justify your answer.

2. Let X be a reflexive normed space. Show that X is complete. (Hint: use the fact that the dual space of any normed space is complete.)

3. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Let $1 < p, q < \infty$ be real numbers satisfying $1/p + 1/q = 1$. Let $M \subset L^p(X, \mathcal{A}, \mu)$ be a subspace and let $\Phi : M \to \mathbb{R}$ be a continuous linear functional. Show that there is a function $h \in L^q(X, \mathcal{A}, \mu)$ with the property that

$$\int_X f h \, d\mu = \Phi(f)$$

for all $f \in M$.