UNIVERSITY OF MANITOBA
DEPARTMENT OF MATHEMATICS

Graduate Comprehensive Exam in Topology

May 6, 2019

10:00 to 16:00 (NO EXTRA TIME PERMITTED)

Examiners: A. Clay
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D. Krepski

INSTRUCTIONS:

You have **six** hours to complete the exam.

The exam consists of six (6) pages, including this cover page.

Answer all seven (7) of the questions in Part A, which is worth a total of 32 marks, distributed according to the following table:

Question	Q1	Q2	Q3	Q4	Q5	Q6	Q7
Value	4	4	5	3	6	5	5

For each of Part B and Part C you have a choice of questions. Answer any three (3) of the four (4) 10 mark questions in each part. Parts B and C are worth a total of 30 marks each.

For either Part B or Part C, you may attempt all four questions in that Part; however, only three answers will be evaluated. **If you submit responses to all four questions appearing in that part, clearly indicate which responses are to be evaluated**. In the absence of any other indication, the first three responses will be evaluated according to the order in which they appear.

To pass this exam, you must obtain a score of at least 75% overall.

PART A Short answer questions

Answer all of Questions 1 through 7. The point values for each question are indicated in square brackets [ ].

[4] **Question 1.**
Recall that a property of a topological space X is called *hereditary* if, whenever X has the given property, so do all of its subspaces. Show that the properties T_0, T_1, and T_2 are hereditary.

[4] **Question 2.**
Let $f : S^1 \to \mathbb{R}$ be a continuous function, where S^1 is the set $\{(x, y) \mid x^2 + y^2 = 1\}$. Prove directly that there exists $x_0 \in S^1$ such that $f(x_0) = f(-x_0)$ (Hint: Consider the sets $\{x \in S^1 \mid f(x) > f(-x)\}$ and $\{x \in S^1 \mid f(x) < f(-x)\}$, and apply connectedness).

[5] **Question 3.**
Let X be a topological space.

(a) Define what it means for a subspace $Y \subset X$ to be a connected component.

(b) Show that if X has finitely many connected components, then each component is an open subset of X.

[3] **Question 4.**
Give an example of a metric space (X, d) and an equivalence relation $\sim$ on X such that the quotient topology on $X/\sim$ is not metrizable.

[6] **Question 5.**

(a) Describe a connected topological space X such that $|\pi_1(X, x_0)| = 10$.

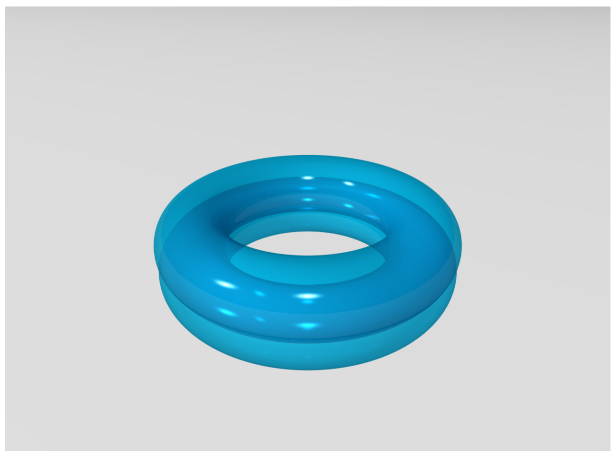
(b) Suppose that $\widetilde{X}$ is a connected space, that $|\pi_1(X, x_0)| = 10$ and $p : \widetilde{X} \to X$ is a covering map. Is it possible that $|p^{-1}(x_0)| = 6$? No credit will be given for an answer lacking justification.

[5] **Question 6.** Let A be a retract of a space X, and let X have the fixed-point property. Show that A has the fixed-point property.

[5] **Question 7.**

(a) Find the fundamental group of the rose of 3 circles.

(b) Let X be made of two intersecting tori, one on top of the other (see figure; $X = T_1 \cup T_2$, where T_1 is a horizontal torus with minor radius r, and T_2 is T_1 translated upwards through a vector of length s, $0 < s < 2r$). Find $\pi_1(X, x_0)$. Do NOT use Seifert - van Kampen theorem.



PART B Point-set Topology

Answer 3 of Questions 8 through 11:

[10] **Question 8.**
Let X be a topological space. Let A and B be compact subspaces of X.

(a) Show $A \cup B$ is a compact subspace of X.

(b) Give an example that illustrates that $A \cap B$ need not be compact.

(c) What 'separation axiom' should X satisfy to guarantee that $A \cap B$ is compact? Justify your response by giving a proof.

[10] **Question 9.**
Let X and Y be normal spaces, and $A \subset X$ a closed subspace. For a continuous map $f : A \to Y$, recall that $X \cup_f Y$ is the quotient of the disjoint union $X \cup Y$ by the equivalence relation (generated by) $a \sim f(a)$. Show that $X \cup_f Y$ is normal.

[10] **Question 10.**

(a) State Urysohn's Lemma.

(b) Show that if X is normal then for every two distinct real numbers a and b and for every disjoint closed sets A and B there is a continuous function $f : X \to [a, b]$ such that $f(A) = a$ and $f(B) = b$.

(c) Show that if X is normal then for every three pairwise distinct real numbers a, b and c and for every three pairwise disjoint closed sets A, B and C there is a continuous function $f : X \to \mathbb{R}$ such that $f(A) = a$, $f(B) = b$ and $f(C) = c$.

[10] **Question 11.**

(a) State Tychonoff Theorem.

(b) Prove Tychonoff Theorem for the case of finitely many spaces.

PART C Algebraic Topology

Answer 3 of Questions 12 through 15:

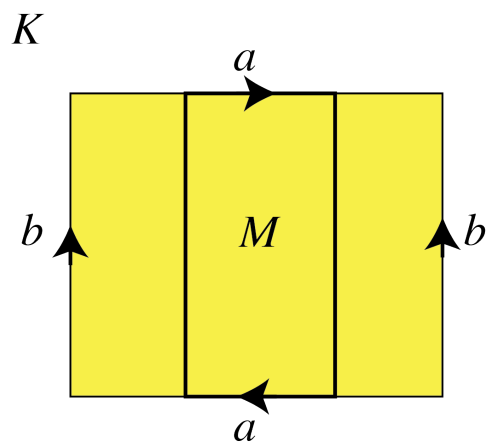
[10] **Question 12.**
For a topological space X, show that the following two conditions are equivalent:

1. Every continuous map $S^1 \to X$ is homotopic to a constant map.
2. Every continuous map $S^1 \to X$ extends to a continuous map $D^2 \to X$, where D^2 is the unit disk.

[10] **Question 13.**

(a) View the Möbius band M as a subspace of the Klein bottle K (see the figure below). Show M is not a retract of K.

(b) Show that there is no retraction from the Möbius band to its boundary circle.



[10] **Question 14.**

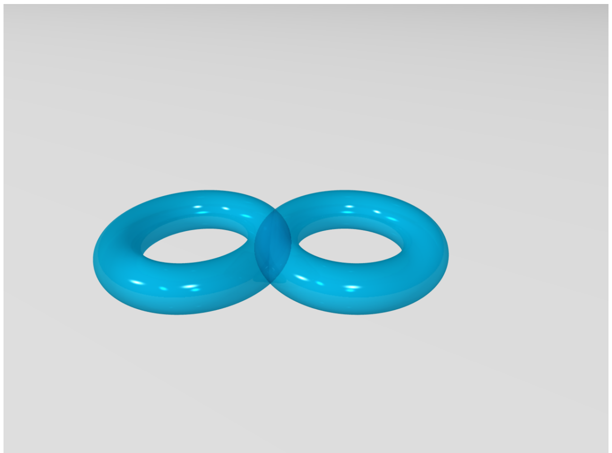
(a) State the definition of a covering space.

(b) Prove that the sphere S^2 is not an n-sheeted cover of a Hausdorff space for every finite $n \geq 3$.

[10] **Question 15.**

(a) State Seifert van Kampen theorem.

(b) Let X be made of two intersecting tori, as in the figure below ($X = T_1 \cup T_2$, where T_1 is a horizontal torus, and T_2 is T_1 translated horizontally as illustrated). Find $\pi_1(X, x_0)$.



END OF EXAM