UNIVERSITY OF MANITOBA
DEPARTMENT OF MATHEMATICS

Graduate Comprehensive Exam in Algebra

Thursday, January 31, 2019. 10:00am–4:00pm

Examiners: T. Kucera (coordinator), S. Sankaran, Yang Zhang

INSTRUCTIONS:

- You have **six hours** to complete the exam.
- No textbooks, notes, calculators, cell-phones, or other devices are permitted at the exam.
- Please write all your answers in the exam booklets provided. Try to keep your solutions to Part A together in one booklet. Number your booklets in order, and if you continue your work on a question in a different booklet, make this clear to the examiners.
- This exam has three parts.

  Part A Part A consists of 10 *short answer* questions worth 2 marks each.
  Attempt all questions in Part A.

  Part B Answer any six of the ten questions in Part B.
  Each question is worth 5 marks.

  Part C Answer any four of the seven questions in Part C.
  Each question is worth 10 marks.

- You may attempt as many questions as you like in Parts B and C, **but if you attempt more than the required number of questions, you must clearly indicate which answers you want us to mark**. In the absence of any other indication, we will mark your solutions in the order that they are presented in your examination booklets.
- In order to pass this exam, you must obtain an overall score of at least 75% (67.5/90).

Be sure to keep in mind that:

- Unless stated otherwise, "vector space" is assumed to include both finite dimensional and infinite dimensional vector spaces.
- Unless stated otherwise, groups can be finite or infinite, and are not assumed to be abelian.
- Unless stated otherwise, "ring" is assumed to mean a ring with unit.
- Unless stated otherwise, "ring" includes both commutative and non-commutative rings.
- Fields may be finite or infinite, of any characteristic.
- Permutations are assumed to act on their arguments from the left, as in usual functional notation, and cycles are displayed in left-to-right order, so if $\sigma = (1, 2, 3)$ then we write $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$, and $\sigma\tau$ is defined by $(\sigma\tau(x) = \sigma(\tau(x))$.

  $\mathcal{S}_n$ denotes the group of permutations of the set $\{ 1, \ldots, n \}$.

PART A *Answer all the questions in this part; each question is worth 2 marks*
All the answers in this part should be short: usually **one to four** sentences.

A1. Let A be an $m \times n$ matrix of rank r over a field $\mathbb{F}$. State the inequalities between r and m, n. Express the dimensions of the row space, the column space, and the null space of A in terms of m, n, and r.

A2. Let C^∞ be the vector space of all infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$, and $\mathsf{D} : C^\infty \to C^\infty$ the linear operator of differentiation. Find all the eigenvalues of D and a non-zero eigenvector belonging to each.

A3. Define what it means for a subnormal series in a group G to be a composition series.

A4. Explain why a non-abelian group must have at least 6 elements, and state an example of a group of order 6 which is not abelian.

A5. Let R be a commutative ring. Prove that every prime element of R is irreducible.

A6. State an example of an ideal of $\mathbb{Z}[x, y]$ which is prime but not maximal, and explain in no more than a couple of sentences why it is prime.

A7. Define *separable polynomial*, and give an example of a field $\mathbb{K}$ and an irreducible polynomial over $\mathbb{K}$ which is not separable.

A8. If $\mathbb{F}$ is a field and $p(x)$ a non-constant polynomial over $\mathbb{F}$, give (without proof) an explicit construction of a finite extension of $\mathbb{F}$ in which $p(x)$ has a root.

A9. Let M be a left R-module. Define "M is flat".

A10. If m and n are relatively prime integers, show that the tensor product of abelian groups $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ is 0.

END OF PART A

PART B *Attempt 6 questions of the 10 questions in this part. If you attempt more than 6 questions, you must clearly indicate which answers you want us to mark; otherwise the first 6 responses in order will be marked* Each question in this part is worth 5 marks.

Question B1. Find a diagonal matrix D similar to $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and an invertible matrix P such that $D = PAP^{-1}$.

Question B2. Suppose that A and B are 3×3 matrices over $\mathbb{C}$ with the same characteristic polynomial $\chi(x)$ and with the same minimal polynomial $\mu(x)$. Prove that A and B are similar.

Question B3. Let G be a simple group of order 168. Prove that there is a non-trivial homomorphism from G to $\mathcal{S}_8$.

Question B4. A subgroup H of the group G is *fully invariant* if for every endomorphism ψ of G, $\psi[H] \subseteq H$. Prove that every fully invariant subgroup of G is normal, and that in partiular the commutator subgroup of G is fully invariant.

Question B5. [There are two possible ways of defining "right noetherian ring". This exercise asks you to verify that the two definitions are equivalent.]

Let R be a ring. Prove that the following are equivalent:

(a) Every right ideal of R is finitely generated.

(b) R satisfies the *ascending chain condition* on right ideals: that is, if $(I_n)_{n\in\omega}$ are right ideals of R such that for all $m < n \in \omega$, $I_m \subseteq I_n$, then there is $N \in \omega$ such that for all $n \geq N$, $I_n = I_N$.

Question B6. Suppose that R is a commutative ring, I is a proper ideal of R, and S is a multiplicatively closed subset of R such that $I \cap S = \emptyset$. Prove that there is a prime ideal P of R, $I \subseteq P \subseteq R \setminus S$.

Question B7. Find the Galois group of $x^4 - 2$ over GF_3.

Question B8. Let p be a prime number and $\mathbb{K}$ a field of characteristic p. Prove that in the rational function field $\mathbb{K}(x)$, x does not have a p-th root.

Question B9. Let M be a commutative ring and M_R a noetherian R-module. Prove that $R/\operatorname{ann}(M)$ is a noetherian ring.

Question B10. Let N be a right R-module. Prove that the functor

$$\mathrm{Hom}(N, —) : \mathrm{Mod}_R \to \mathbf{Ab}$$

is left exact.

END OF PART B

PART C *Attempt 4 questions of the 7 questions in this part. If you attempt more than 4 questions, you must clearly indicate which answers you want us to mark; otherwise the first 4 responses in order will be marked* Each question in this part is worth 10 marks.

Question C1. Define the concepts of linear independence, generating or spanning set, basis, and dimension in a vector space over a field $\mathbb{F}$.

Prove that a maximal linearly independent set is a spanning set, and that a minimal spanning set is linearly independent.

Prove that every vector space over $\mathbb{F}$ has a basis.

Question C2. Prove *Schreier's Theorem*: Any two subnormal series of the same group have equivalent refinements.

You will need to use the so-called "Third Isomorphism Theorem for Groups", also known as the "Zassenhaus Lemma" or the "Butterfly Lemma":

Let G_1 and G_2 be subgroups of a group G, and H_1, H_2 normal subgroups of G_1, G_2 respectively. Then

$$(G_1 \cap G_2)H_1/(G_1 \cap H_2)H_1 \cong (G_1 \cap G_2)H_2/(H_1 \cap G_2)H_2$$

Question C3. Let $n \geq 3$ be an odd integer, and consider the ring

$$R_n := \mathbb{Z}[\sqrt{-n}] = \left\{a + bi\sqrt{n} \,:\, a, b \in \mathbb{Z}\right\} \subset \mathbb{C}$$

(a) Define $\mathsf{N}(a + bi\sqrt{n}) = a^2 + nb^2)$.

Prove that for α, $\beta \in R_n$, $\mathsf{N}(\alpha\beta) = \mathsf{N}(\alpha)\mathsf{N}(\beta)$.

(b) Find all the units of R_n.

(c) Show that 2 is irreducible but not prime in R_n.

(d) Show that R_n is not a UFD.

Question C4. A commutative ring R is called *local* if it has a unique maximal ideal.

Let D be an integral domain, $Q(D)$ the quotient field of D, and M a maximal ideal of D. Let

$$D_{(M)} = \left\{\frac{a}{b} \in Q(D) \,:\, b \in D \setminus M\right\}$$

(a) Show that $D_{(M)}$ is a subring of Q containing D;

(b) Show that $D_{(M)}$ is a local ring with maximal ideal $MD_{(M)}$;

(c) Show that $u \in D_{(M)}$ is a unit of $D_{(M)}$ iff $u \notin MD_{(M)}$.

Question C5. Let $\mathbb{F}$ be a field and $\mathbb{F}^{\times}$ be the multiplicative group of $\mathbb{F}$, defined on $\mathbb{F} \setminus \{0\}$.

Show that any finite subgroup of $\mathbb{F}^{\times}$ is cyclic.

Question C6. Let ζ be a primitive 13th root of unity.

(a) Show that $\mathbb{Q}(\zeta)$ is Galois over $\mathbb{Q}$, and explicitly construct the Galois group. (Do not just quote a theorem; justify your answer.)

(b) Find all the proper subfields of $\mathbb{Q}(\zeta)$.

Question C7. Prove that a left R-module M is injective iff every homomorphism $f : {}_R I \to M$ from a left ideal of R can be extended to a homomorphism $\bar{f} : {}_R R \to M$.

END OF EXAM