Analysis comprehensive examination
Department of Mathematics

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This examination consists of three parts.

- Part A covers the core material. It has 7 questions worth 10 points each, and you must attempt all questions in this part for a total possible score of 70 points.
- Part B covers the specialized material on abstract measure and integration. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.
- Part C covers the specialized material on basic functional analysis. It has 3 questions worth 15 points each, of which you must attempt 2, for a total possible score of 30 points.

If you attempt more than the required number of questions in Part B or C, you must clearly indicate which questions are to be graded. If it is not clearly indicated, solutions to those appearing first in the booklet will be graded.

You need to achieve at least 97.5 points, which is 75% of the total possible 130 points, in order to pass the examination.

The total time of the examination is six hours. No books, notes, calculators or aids are allowed during the exam.

Part A

Solve *all* of the following problems.

Problem 1.

(a) (6 points) Let E be an open, bounded, connected set in $\mathbb{R}^3$. Let (a, b) be a non-empty open interval in $\mathbb{R}$. Let $\rho(t, x, y, z)$ be a function on $(a, b) \times E$ with continuous second derivatives. Assume that ρ satisfies the differential equation

$$\frac{\partial \rho}{\partial t} = \triangle \rho$$

where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Now let $[c, d] \subseteq (a, b)$ and let $D \subset E$ be a compact region bounded by a smooth surface S. Let

$$m(t) = \iiint_D \rho(t, x, y, z) dV.$$

Show that on (c, d) we have

$$\frac{dm}{dt} = \iint_S \nabla \rho \cdot d\mathbf{S}$$

where $\nabla \rho$ denotes gradient of ρ with respect to (x, y, z), and the integral is the surface integral of the vector field $\nabla \rho$ over S.

(b) (4 points) Let S_1 and S_2 be two oriented smooth surfaces in $\mathbb{R}^3$ bounded by the same smooth curve C. Let $\mathbf{F}$ be a vector field with continuous partial derivatives on $\mathbb{R}^3$. Show that

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \pm \iint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Problem 2. Let $\mathcal{H}$ be a Hilbert space and let $S \subset \mathcal{H}$ be its unit sphere, that is

$$S = \{x \in \mathcal{H} : \|x\| = 1\}.$$

Let $u, v \in S$ be distinct points and let $\Gamma \subset \mathcal{H}$ denote the straight-line segment joining u to v. Show that the midpoint of Γ does not belong to S.

Problem 3. Let (X, d) and (Y, ρ) be metric spaces. Assume that (X, d) is compact. Let $f : X \to Y$ be a continuous mapping. Show that f is uniformly continuous.

Problem 4.

(a) (3 points) State Liouville's theorem for complex analytic functions.

(b) (7 points) Let $f(z)$ be a complex analytic function on the complex plane $\mathbb{C}$. Assume that $a_1, \ldots, a_n \in \mathbb{C}$ are simple zeros of f. Assume that there is a constant M such that

$$|f(z)| \leq M|z|^n$$

for all $z \in \mathbb{C}$. Show that f is a polynomial of degree n. *Hint:* what can you say about

$$\left| \frac{z^n}{(z - a_1) \cdots (z - a_n)} \right|$$

as $|z| \to \infty$?

Problem 5. For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be a function for which there is $L_n \in \mathbb{R}$ with the property that

$$\lim_{x \to 0} f_n(x) = L_n.$$

Assume that the sequence of functions $\{f_n\}$ converges uniformly on $\mathbb{R}$ to some other function $g : \mathbb{R} \to \mathbb{R}$. Find a necessary *and* sufficient condition on the sequence $\{L_n\}$ for the function g to have a limit at $x = 0$. Prove your claim.

Problem 6.

(a) (5 points) Let f be a complex analytic function on an open connected set $U \subset \mathbb{C}$. Let u be the real part of f. Show that u is harmonic.

(b) (5 points) Let $H = \{z = x + iy \in \mathbb{C} : x > 0\}$. Define a function u on H as

$$u(x, y) = \arctan(y/x) + e^x \sin y.$$

Find a harmonic function v on H such that $f(z) = u(x, y) + iv(x, y)$ is complex analytic on H.

Problem 7.

(a) (6 points) Let $f : [0, 1] \to \mathbb{R}$ be an increasing function. Show that f has only countably many points of discontinuity.

(b) (4 points) Let $f : [0, 1] \to \mathbb{R}$ be a function of bounded variation. Show that f has only countably many points of discontinuity.

Part B

Solve 2 out of the following 3 problems.

Problem 1. Let X be a measurable space. Let f and g be real-valued measurable functions defined on X.

(a) (5 points) Show that the sum function $f + g$ is measurable on X.

(b) (10 points) Show that the product function fg is measurable on X.

Problem 2. Apply an appropriate convergence theorem for integrals to show the following theorem:

If f is an integrable function over the measure space (X, μ), then for each $\epsilon > 0$ there is $\delta > 0$ such that $|\int_A f d\mu| < \epsilon$ whenever A is a measurable subset with $\mu(A) < \delta$.

Problem 3. Consider the Lebesgue measure spaces $(\mathbb{R}, \mathrm{m})$ and $(\mathbb{R}^2, \mathrm{m} \times \mathrm{m})$. Show that if E is a measurable subset of $\mathbb{R}$, then

$$D = \{(x, y) \in \mathbb{R}^2 : x - y \in E\}$$

is a measurable subset of $\mathbb{R}^2$. *Hint*: Argue step by step. First consider E to be open, then of type G_δ, and then of measure zero. You may use the fact that $\mathrm{m} \times \mathrm{m}$ is complete.

Part C

Solve 2 out of the following 3 problems.

Problem 1. Let X be a normed space. Give the definition of the canonical embedding of X into its second dual, and show that it is contractive.

Problem 2. Let X be a normed space and let $Y \subset X$ be a subspace. Show that there is a contractive and surjective linear map $R : X^* \to Y^*$. *Hint* : use the Hahn-Banach theorem.

Problem 3. Let X be a Banach space. For each $n \in \mathbb{N}$, let $T_n : X \to X$ be a bounded linear operator. Assume that for each $x \in X$, the sequence $\{T_n x\}$ is convergent in X. Show that the sequence $\{T_n\}$ is bounded with respect to the operator norm.