University of Manitoba
Department of Mathematics

Ph.D. Comprehensive Examinations
Regulations and Syllabi

Date: June 14, 2018
Ph.D. Comprehensive Examinations

A. Regulations

1. From the Department of Mathematics Supplemental Regulations for the Ph.D. Program:
   “The candidacy examination in the Ph.D. program in Mathematics consists of two separate written comprehensive examinations chosen from the following areas: a) Algebra; b) Analysis; c) Combinatorics; d) Computational Mathematics; e) Differential Equations; f) Topology. Regulations governing these examinations and the latest syllabi on which these examinations are based are described in the document "Ph.D. Comprehensive Examinations REGULATIONS and SYLLABI" approved by the Department Council, and are available from the Associate Head (Graduate Studies). At least one of the examinations must be in Algebra or Analysis. The Graduate Studies Committee arranges comprehensive examinations three times a year, normally in January, April and September. The student must register to write a comprehensive examination by sending a request to the Associate Head (Graduate Studies) by February 1 if the examination is given in April, by July 1 if the examination is given in September and by November 1 if the examination is given in January of the following year. The choice of areas must be approved by the student’s advisor.
   The standard of pass shall be given on the question sheet of each examination. A maximum of two attempts on each comprehensive examination is allowed. A student who fails a comprehensive examination in any area twice shall be required to withdraw from the Ph.D. program in Mathematics.”

2. Once a student has made a formal request for an examination (either in writing or by email), he/she is obligated to write it (except that a request can be withdrawn before March 1 if the examination is given in April, before August 1 if the examination is given in September and before Dec 1 if the examination is given in January of the following year). Absence from the examination on medical or compassionate grounds will be excused according to the same policies that apply to final examinations in the Faculty of Science. Any other delay or deferral of the examination shall be considered by the Graduate Studies Committee only upon receipt of a written request from a student’s advisor (this request must be received by the Graduate Studies Committee before the examination) outlining the specific reasons for the delay or deferral.

3. Each Examining Committee shall consist of at least three persons. One member of each committee will be designated as Coordinator and will be responsible for communicating with the students, with the Graduate Studies Committee through the Associate Head (Graduate Studies) and with the Department office (through the Administrative Assistant).
   The Examining Committee sets the questions for the examination as it sees fit. Once the examination has reached its final form it is the responsibility of every committee member to read the entire examination and ensure that it is consistent with the syllabus and with the general level of difficulty expected.
   The Coordinator is responsible for ensuring that all the regulations in this document are adhered to.

4. This regulation applies to all comprehensive examinations unless the syllabus for an examination explicitly overrides it.

   Style of the examination:
   The examination shall consist of up to three parts A, B and C.
a) Part A shall consist of $N_1$ mandatory questions, worth a total of $W_1$ marks (individual marks for each question may or may not be the same). Here, $N_1$ is a positive integer and $W_1$ is positive.
b) Part B (if any) shall provide a choice of $N_2$ questions worth $\omega_2$ marks each with the instruction that the student is allowed to attempt $M_2$ out of $N_2$ questions ($M_2$ shall be strictly less than $N_2$) with a clear indication of which $M_2$ questions he/she wants to have marked (and so the maximum marks that a student can obtain in this part shall be $W_2 := \omega_2 M_2$).
c) Part C (if any) shall provide a choice of $N_3$ questions worth $\omega_3$ marks each with the instruction that the student is allowed to attempt $M_3$ out of $N_3$ questions ($M_3$ shall be strictly less than $N_3$) with a clear indication of which $M_3$ questions he/she wants to have marked (and so the maximum marks that a student can obtain in this part shall be $W_3 := \omega_3 M_3$).

It is up to the Examining Committee to decide how many questions will be in each part and how many marks will each part be worth (i.e., there are no restrictions on the values of $N_i$, $M_i$ and $W_i$), and there is no obligation to follow the format/style of past examinations. However, students shall be given 6 hours to complete each examination. This time limit is intentionally generous. The expectation is that the examination can be successfully finished in less time (sometimes significantly less). Students shall be given all the questions of the entire examination at the beginning of the examination.

**Passing criteria:** In order to pass the examination the student needs to obtain at least 75% overall, i.e., the minimum passing mark shall be $\frac{3}{4}(W_1 + W_2 + W_3)$.
the Examining Committee shall use the “pass criteria” already set and indicated on the examination paper to determine if the student passes or fails the examination. The Coordinator shall write a report containing the mark decided upon by the Examining Committee, but not marks by individual examiners, of each question and comments (if any). The report must clearly indicate the result of the examination (“pass” or “fail”). It cannot contain statements of the sort “passed/failed notwithstanding”. The report shall be given to the Associate Head (Graduate Studies) who will forward it the candidate as well as the Department for storage.

9. During the appeal period, neither the Coordinator nor the Committee shall communicate with the students, their advisors or anybody else other than the Associate Head (Graduate Studies) and the Department Head on any matter pertaining to the examination unless the Associate Head or the Department Head instructs otherwise (in writing).

10. The Associate Head (Graduate Studies) or Department Head shall announce the results formally to each of the students by means of a letter on Department letterhead. A copy of each letter must be provided to the Department office for the student’s files. In the case of a failure, the letter shall inform the student of the right to appeal the outcome (with a reminder about the internal two week deadline). In the case of a first failure, the letter shall inform the student of the possibility to rewrite the examination; and in the case of a second failure on a given examination that this result means that the student will be required by the Faculty of Graduate Studies to withdraw from the Ph.D. program.

11. In the case that a first failure has been reported, and after the appeal period is over, it is the responsibility of the Coordinator to ensure that the student receives general feedback on the reasons for the failure and some guidance in study and preparation for the second attempt.

12. After the result of the examination has been announced, the student may view a copy of his/her examination paper in the Department office, under supervision. Students shall not be allowed to make any copies of any of the examination papers at any time.

13. Appeals. If a student wishes to appeal the result of his/her examination, a formal appeal in writing must be addressed to and received by the Department Head within 10 (ten) working days of the announcement of the result to the student. Appeals based solely on disagreement of the allocation of marks for one or more questions will be automatically rejected.

B. Advice to students preparing for Ph.D. Comprehensive Examinations

1. A Graduate Comprehensive Examination in the Department of Mathematics is an examination of material that is normally taught in an undergraduate honours program. For greater clarity, any material that is covered by the syllabus may appear on a comprehensive examination irrespectively of whether or not it has been (or is) taught in undergraduate or graduate courses at the University of Manitoba or any other institution. Additionally, since it is impractical (if not impossible) to list all possible definitions/concepts in the syllabus, it should be understood that topics related to those in the syllabus are also covered. For example, if a section of a book is covered by a syllabus, and a topic is discussed in this section, then this topic may be covered by the examination even if that specific concept is not explicitly mentioned in the syllabus. It goes without saying that you should also know all the basic material that is a prerequisite for the topics that are covered by the syllabus. Even though comprehensive examinations are based on undergraduate material, they are being written by Ph.D. students, and so a more sophisticated level of understanding and presentation may be expected than might normally be demanded in relevant undergraduate courses.
2. Copies of old comprehensive examinations may be available from the Department office or from the Department website. However, you should have ABSOLUTELY NO EXPECTATIONS that future examinations will be identical or even remotely similar to any particular previous examination. Do not base your expectations for the examination on any particular previous examination.

3. Do not limit your study and preparation to one reference; reviewing all references provided with the syllabus will provide a much better preparation for different styles of questions.

4. Texts and old examinations from relevant undergraduate courses may be useful as a study supplement, but students should master the material in the syllabus of the comprehensive examinations which follow.

5. Do not expect any Faculty member to “teach” you the subject matter of the examination (outside the regular coursework). You are admitted to our Ph.D. program on the basis that you are (mostly) prepared for work at this level, including being prepared (or having the mathematical maturity to be able to prepare) for the Comprehensive Examinations. You should be able to prepare for these examinations by reviewing of your earlier studies, and by self-studying of any subjects that you have missed.

6. Faculty members may be able to advise you on relevant study plans, be helpful with specific difficulties that you may have, and so on. The Associate Head (Graduate Studies) can help clarify the regulations; the Coordinator of the Examining Committee may help you interpret the details of the Examination Syllabus; and members of Faculty working in the general area of the Examination are often available for help with specific questions.

7. Expect a mixture of theoretical and practical problems, sometimes in the same question. Classifying questions as theoretical or practical is in many cases arbitrary, and a comprehensive examination in any case may (and often will) cover the entire range of styles of questions suitable to the subject area.

8. Corrective action can be taken at any time if mistakes in grading and/or questions are found, even after expiration of the appeal period.
Algebra Comprehensive Examination Syllabus

1. Linear Algebra.
   - Vector spaces over $\mathbb{R}$, $\mathbb{C}$, finite fields and general fields.
   - Subspaces. Linear independence, generating (spanning) sets, bases. Dimension. Infinite dimensional vector spaces. Direct sums. Every vector space has a basis (application of Zorn’s Lemma or equivalent).
   - Linear operators on finite and infinite dimensional spaces.
   - Linear functionals. Duals of both finite and infinite dimensional spaces.
   - Jordan canonical form.
   - Bilinear forms and their representations. Quadratic forms. Positive definite and positive semidefinite bilinear forms.

2. Group Theory.
   - Semigroups, monoids, groups. Left and right inverses, units, idempotents. Order of an element of a group, order of a group. Cyclic groups. Permutation groups, Cayley’s representation theorem. Dihedral groups, matrix groups, examples of symmetry groups. Group actions, stabilizer, orbit.
   - Subgroups, cosets, normal subgroups. Lagrange’s theorem.
   - Centre, centralizer, normalizer, conjugate subgroups. Class equation.
   - Sylow Theorems. $p$-groups, Sylow $p$-subgroups. Applications of the Sylow theorems. Nilpotent groups.
   - Direct product (internal and external) of groups.
   - Free abelian groups. The structure of finitely generated abelian groups, of finite abelian groups.
3. Ring Theory.

- Embedding an integral domain in a field (quotient field construction).
- Unique Factorization Domains, Principal Ideal Domains, Euclidean Domains and the implications between them. Factorization theory in integral domains.
- Chain conditions (ascending chain condition, descending chain condition), noetherian and artinian rings. Existence of maximal ideals and prime ideals (application of Zorn’s Lemma or equivalent).


- Finite fields, order of finite fields. Prime subfields.
- Quadratic extensions, straightedge and compass constructions, constructible numbers.
- Determining the Galois group of a polynomial. Galois group of the splitting field of a polynomial and permutations of the roots of the polynomial.

5. Module Theory.

- Product, direct sum, tensor product and their characterization by universal properties. The Hom functor. Short exact sequences and preservation of exactness.
Injective, free, projective, and flat modules.

Suggested reading list:

- Primary references:

  
  [Chapters 1–9, 10.1–10.5, 10.8, 10.11, 11.1–11.5, Appendix 1]

  
  [Sections 1.1, 1.2, 2.1–2.5, 4.1, 4.2, 4.4–4.8, 7]

  The rest of this book is well beyond the scope of this syllabus.

- For an in-depth review of Linear Algebra and Galois Theory:


- Additional reading:


  (Not *Abstract Algebra*, which is more elementary)


  (Not *Undergraduate Algebra*, which is more elementary)

The exam questions are not selected from any one reference or references.
Analysis Comprehensive Examination Syllabus

The analysis comprehensive examination consists of two units: UNIT ONE, which is for the core material and UNIT TWO, which is for the specialized material. For UNIT TWO, each student shall choose 2 of the 3 subsections II.(a), II.(b) and II.(c) below that will be covered by the examination. The student must declare his/her choices when registering for the examination. UNIT TWO of the examination will only cover the two subsections that the student has previously chosen.

I. The core material

Students are expected to know all basic mathematical analysis/calculus (see, e.g., [11, Chapters 1-9] and corresponding sections from [10]). In addition, the following topics are required:

- Functions of bounded variation: Monotonic functions, Total variation, Functions of bounded variation expressed as the difference of increasing functions.
- Riemann and Riemann-Stieltjes integral: Definition and elementary properties, Integration by parts, Change of variable, Step functions as integrators, Reduction of Riemann-Stieltjes integral to a finite sum, Fundamental theorems of integral calculus.
- Sequences of functions: Pointwise and uniform convergence, Uniform convergence and continuity, Cauchy condition for uniform convergence, Uniform convergence of infinite series of functions, Uniform convergence and Riemann-Stieltjes integration, Uniform convergence of improper integrals, Uniform convergence and differentiation, Sufficient conditions for uniform convergence of a series (Weierstrass M-test, Dirichlet’s test), Weierstrass approximation theorem.
- Inverse function and implicit function theorems, Theorems of Green, Gauss and Stokes.
- Fourier series: Orthogonal systems of functions, Theorem on best approximations, Fourier series of a function relative to an orthonormal system, Properties of the Fourier coefficients, Riesz-Fischer theorem.
- Basics of Hilbert spaces.
- Lebesgue Measure
  - Lebesgue outer measure, Measurable sets and Lebesgue measure, $G_δ$ and $F_σ$ sets, Existence of a non-measurable set, Measurable functions, Egorov’s Theorem.
- Lebesgue Integral
  - Definition and properties of Lebesgue integral, Fatou’s Lemma, Monotone convergence theorem, Dominated convergence theorem, Convergence in measure.
- Differentiation and Integration
  - Vitali’s cover and Vitali’s theorem, Integral of the derivative of an increasing function, Differentiation of the indefinite integral, Absolutely continuous functions.
- Basic properties of complex numbers and analytic functions, Elementary analytic functions, Entire Functions, Cauchy-Riemann Theorem, Inverse Function Theorem.
- Contour Integrals, Cauchy’s Theorem, Cauchy’s Integral Formula, Cauchy’s Inequalities, Liouville’s Theorem, Fundamental Theorem of Algebra, Morera’s Theorem, Maximum Modulus Theorem, Schwarz Lemma.
Residue Theorem and evaluation of integrals by using the Residue Theorem.

Identity Theorem, Rouche’s Theorem and Principal of the Argument Theorem, Hurwitz’s theorem, Open Mapping Theorem for analytic functions.

The topics listed above may be found in [10, Sections 6.1-6.7, 7.1-7.9, 7.19, 7.20, 9.1-9.11, 10.13, 11.1-11.6, 11.15], [14, Chapters 1-5 and Section 10.8] and [13, Chapters 1-6].

II. The specialized material

(a) Basic Functional Analysis

Normed Spaces
Definition of normed and Banach spaces, Subspaces, Quotient spaces, Linear operators and functionals, Equivalent conditions for continuity of linear operators, Spaces of bounded linear operators, Hahn-Banach theorem, Embedding of a normed space into its second dual, Reflexive Banach spaces, Closed graph theorem, Open mapping theorem, Baire category in metric spaces, Principle of uniform boundedness.

$L^p$-Spaces
The Minkowski and Holder inequalities, Completeness, Dual spaces of $L^p$-spaces (Riesz representation theorem).

The topics listed above in this subsection may be found in [14, Chapters 6 and 10].

(b) Abstract Measure and Integration

Measure spaces, Measurable functions, Simple functions, Definition of abstract integral, linearity with respect to integrand and additivity with respect to domain of integration, Convergence theorems (Monotone, Dominated, Fatou’s lemma), Signed measures, Complex measures, Theorem of Hahn for signed measures, Absolute continuity and singularity for measures, total variation of a measure, Radon Nikodym theorem, Lebesgue decomposition theorem, Outer measures, Measures derived from outer measures, Extension of a measure on an algebra (semi-algebra) to a measure on a $\sigma$-algebra, Caratheodory theorem, Product measures and theorems of Fubini and Tonneli.

The topics listed above in this subsection may be found in [14, Chapters 11 and 12].

(c) Advanced Complex Analysis

Conformal mappings and linear fractional transformations, Spaces of analytic functions, Equicontinuity, Uniform convergence on compact sets, Completeness of families of analytic and meromorphic functions, Normal families, Montel’s theorem, Riemann mapping theorem, Harmonic functions, Mean Value Property for harmonic functions, Maximum Modulus Theorems for harmonic functions, Solution of the Dirichlet problem, Harnack’s inequality and Harnack’s Theorem.

The topics listed above in this subsection may be found in [12, Chapters III, VII and X].

References:


Combinatorics Comprehensive Examination Syllabus

The combinatorics comprehensive exam covers three mandatory areas: core combinatorial concepts, graph theory, and elementary design theory. For study purposes, the three areas may constitute an approximate 40:35:25 distribution of available marks.

Unless otherwise noted, the student is expected to know proofs of named theorems and of main theorems of each topic covered. The student is assumed to have a working knowledge of modular arithmetic, finite fields, and mathematical induction.

Core combinatorial concepts

**Basic counting:** Counting rules (product and sum rules), permutations, r-permutations, subsets, r-combinations, binomial coefficients, probability, sampling with replacement, occupancy problems (e.g., indistinguishable balls into distinguishable cells), Stirling numbers of the second kind, multinomial coefficients, binomial expansion, generating permutations and combinations, Gray codes, algorithms and complexity, pigeonhole principle, partitions of integers and Ferrers diagrams.

**Relations:** Binary relations, order (linear, partial), lexicographic order, linear extensions, chains, antichains, Sperner’s lemma, Dilworth’s theorem, interval orders, lattices, distributive and modular lattices, complementation in lattices, Boolean lattices.

**Generating functions:** Applications to distribution, composition, and partitions of integers, generalized binomial theorem, exponential generating functions, counting permutations, distinguishable balls into indistinguishable cells.

**Recurrence relations:** Fibonacci numbers, derangements, method of characteristic roots, recurrences involving convolutions, Catalan numbers (including counting lattice paths, triangulations), solution by generating functions.

**Inclusion and exclusion:** Principle of inclusion-exclusion, counting derangements, number of objects having exactly \( m \) properties.

References for core concepts


Graph theory

Basic graph theory: Graph, multigraph, labelled graph, degree, degree sequence, handshaking lemma, Havel–Hakimi theorem, isomorphism, subgraph, complete graph, regular graph, bipartite graph, complement, hypercubes, connected, cut vertex, block, bridge, distance, walk, adjacency matrix, counting walks, incidence matrix, trail, path, Dijkstra’s algorithm, circuit, cycle, girth, radius, diameter, centre, eccentricity.

Trees: Their properties, chemical bonds and trees, Cayley’s theorem for labelled trees, Prüfer sequences, minimum spanning trees, Kruskal’s algorithm, Prim’s algorithm.

Directed graphs: Digraph, tournament, strongly connected, transitive tournament, king, Robbin’s theorem for one-way streets.

Cycles and circuits: Eulerian circuits and trails in graphs and digraphs, Fleury’s algorithm, DeBruijn sequences, rotating drum problem, Chinese postman problem, Hamiltonian paths and cycles, Travelling Salesman Problem, Ore’s condition, Dirac’s condition, independent set.

Planar graphs: Euler’s formula for planar graphs, five regular polyhedra, planar duals, Four Colour Theorem (statement only), Kuratowski’s and Wagner’s theorems (statements only).

Colouring: Vertex colourings, chromatic number, Brooks’ theorem, Nordhaus–Gaddum theorem, perfect graph, interval graph, chordal graph, edge colouring, chromatic index, Vizing’s theorem (statement only), König’s theorem (for chromatic index), Ramsey’s theorem, small Ramsey numbers, upper and lower bounds for \( R(k,k) \).


References for graph theory


Elementary design theory

BIBDs, dual and complement of a design, latin squares, problem of 36 officers, MOLS, latin squares over finite fields, finite projective planes, homogeneous coordinates, incidence matrices, $t$-designs, derived and residual designs, Hadamard matrices, resolvable designs, Kirkman's school girl problem, Steiner triple systems, Fisher's inequality, and statements of Bruck-Ryser-Chowla and Bruck-Ryser theorems.

References for design theory

Computational Mathematics Comprehensive Examination Syllabus

The Computational Mathematics comprehensive examination consists of 2 (two) Units X and Y.

- Unit X shall be based on the material from section I (Introductory Numerical Analysis).
- Unit Y shall be based on the material from 1 (one) of sections II (Numerical PDEs) or III (Approximation Theory).

The choice of Unit Y shall be declared by the student at the time of registration for the Computational Mathematics comprehensive examination.

I. Introductory Numerical Analysis

▷ Computer arithmetic and floating-point numbers, relative and absolute errors. Round-off-errors and their propagation.
▷ Polynomial Interpolation: Lagrange interpolation, Divided differences, Natural and clamped cubic spline interpolation.
▷ Numerical differentiation and integration: Richardson's extrapolation, trapezoid and Simpson’s rules, Newton-Cotes formulas, Composite numerical integration, Gaussian quadratures.
▷ Least squares approximation, orthogonal polynomials approximation (Legendre and Chebyshev), and trigonometric polynomial approximation.
▷ Direct methods for solving linear systems: Gaussian elimination, Pivoting methods, Matrix factorization and special types of matrices.

References:
The material covered in this section may be found in Chapter 2 of [1], and Sections 1.2, 2.1-2.4, 3.1, 3.3, 3.5, 4.1-4.4, 4.7, 5.2-5.4, 5.6, 5.9-5.10, 6.1, 6.2, 6.5, 6.6, 7.3-7.5, 8.1-8.3, 8.5, 9.3, 10.1, 10.2, 11.1-11.4. of [2].

II. Numerical Analysis of PDEs

- Explicit and implicit finite difference schemes for parabolic partial differential equations (PDEs) including Euler and Crank Nicholson schemes, convergence, consistency and stability (von Neumann stability, matrix stability).
- Finite difference schemes (second order, fourth order, upwind schemes) for elliptic PDEs, convergence, consistency and stability.
- Finite difference schemes for hyperbolic PDEs, convergence, consistency and stability, CFL condition.
- Finite element methods for elliptic PDEs: 1D and 2D finite elements, local and global interpolation error estimates, $L^2$ and energy norm error estimates, inverse estimates.
- Numerical Linear Algebra: Condition number, classical iterative methods and Krylov subspace methods, preconditioning.

References:
The material covered in this section may be found in Chapters 1 to 7 of [3], Sections 3.1-3.6, 4.1-4.4, 9.1-9.8, 10.1-10.5, 10.7 of [4] and Chapters 1, 3.1-3.5, 4 and 6 of [5].


III. Approximation Theory

- Density, existence, uniqueness
  

- Polynomial approximation
  

- Non-polynomial approximation
  

References:
The material covered in this section may be found in [7] and in Sections 1.1-1.4, 2.1, 2.6, 2.7, 2.9, 3.1-3.6, 4.1, 4.5-4.7, 5.1-5.3, 5.5-5.9, 6.1-6.4, 7.1-7.3, 7.6, 7.7, 8.1-8.6, 10.1-10.3 of [6].

Differential Equations Comprehensive Examination Syllabus

Style of the examination:
The examination shall consist of 2 (two) units carrying equal weights: Unit I (Ordinary Differential Equations) and Unit II (Partial Differential Equations). For \( X \in \{ I, II \} \), unit \( X \) shall consist of up to three parts A, B and C.

a) Part A shall consist of \( N_{1,X} \) mandatory questions, worth a total of \( W_{1,X} \) marks (individual marks for each question may or may not be the same).

b) Part B (if any) shall provide a choice of \( N_{2,X} \) questions worth \( \omega_{2,X} \) marks each with the instruction that the student is allowed to attempt \( M_{2,X} \) out of \( N_{2,X} \) questions (\( M_{2,X} \) shall be strictly less than \( N_{2,X} \)) with a clear indication of which \( M_{2,X} \) questions he/she wants to have marked (and so the maximum marks that a student can obtain in this part shall be \( W_{2,X} := \omega_{2,X} M_{2,X} \)).

c) Part C (if any) shall provide a choice of \( N_{3,X} \) questions worth \( \omega_{3,X} \) marks each with the instruction that the student is allowed to attempt \( M_{3,X} \) out of \( N_{3,X} \) questions (\( M_{3,X} \) shall be strictly less than \( N_{3,X} \)) with a clear indication of which \( M_{3,X} \) questions he/she wants to have marked (and so the maximum marks that a student can obtain in this part shall be \( W_{3,X} := \omega_{3,X} M_{3,X} \)).

It is up to the Examining Committee to decide how many questions will be in each part of each unit and how many marks will each part be worth (i.e., there are no restrictions on the values of \( N_{i,X}, M_{i,X} \) and \( W_{i,X} \)), and there is no obligation to follow the format/style of past examinations.

Passing criteria: In order to pass the examination the student needs to obtain at least 75% overall, i.e., the student will pass the examination if

\[
\frac{\mu_I}{W_{1,I} + W_{2,I} + W_{3,I}} + \frac{\mu_{II}}{W_{1,II} + W_{2,II} + W_{3,II}} \geq \frac{3}{2}.
\]

where \( \mu_I \) and \( \mu_{II} \) are the marks obtained by the student in Units I and II, respectively.

Students are expected to know topics from analysis used in proofs in the suggested references; for example, complete metric spaces, Cauchy sequences, uniform continuity and uniform convergence, Lipschitz functions, etc.

Unit I. Ordinary Differential Equations

▷ General theory of ordinary differential equations

▷ Techniques for scalar first-order equations
Separable and linear equations. Substitution methods: homogeneous and Bernoulli equations; equations of the form \( y' = f(ax + by + c) \). Exact equations and integrating factors.

▷ Scalar \( n \)th-order linear equations
Homogeneous linear equations (principle of superposition; linear independence; Wronskian;
fundamental set of solutions; general solution; solution to an initial value problem). Nonhomogeneous linear equations (general theory; variation of constants – also called variation of parameters). Methods to solve $n^{th}$-order linear equations with constant coefficients (characteristic equation; differential operators and the annihilator method; undetermined coefficients; Laplace transform – unit step, Dirac Delta and periodic functions, convolution). Methods to solve $n^{th}$-order linear equations with non-constant coefficients (reduction of order; Cauchy-Euler equations; series solutions about ordinary points and regular singular points). Limiting behaviour of solutions.

▷ Linear systems of first-order differential equations
Transformation of a scalar $n^{th}$-order linear equation into a system of $n$ scalar first-order differential equations. Homogeneous systems (superposition principle; the vector space of solutions – fundamental set of solutions; fundamental matrix). Autonomous homogeneous systems (diagonalizable case; generalized eigenspace; matrix exponential; trichotomy – stable, unstable and centre subspaces). Nonhomogeneous systems (variation of constants).

▷ Nonlinear systems

▷ Basic bifurcation theory
Normal forms of the saddle-node, transcritical and pitchfork bifurcations. Bifurcation diagrams. Transformation of the normal form of the Hopf bifurcation to polar coordinates. Study of the Hopf bifurcation in the polar plane, with construction of the Poincaré map.

▷ Particular case of scalar 2nd-order equations (planar systems)

Coefficients of the 4th-order Runge-Kutta method and tables of Laplace transforms will be provided, as needed.

Suggested references


Markley [1] is the main reference for Unit I; all material in it is to be known, with the exception of Sections 3.4, 4.5, 5.4, 7.4 and Chapter 6. Nagle, Saff and Snider [2] is a complementary resource with a presentation that is generally not as rigorous as in [1]; in the case where a result is present in both [1] and [2], it is the version in [1] that is used. In Perko [3], use Chapter 1, Sections 2.1 to 2.13, 3.4, 4.2 and 4.4.
Unit II. Partial Differential Equations

▷ **Classification**

▷ **Linear and quasilinear first-order equations**

▷ **Linear second order equations in two independent variables**

▷ **Separation of variables - second order linear equations in two or more variables**
  Cartesian, polar, cylindrical, and spherical coordinates. Nonhomogeneous problems (reduction to a homogeneous problem; steady-state solutions; eigenfunction expansions).

▷ **Sturm-Liouville theory and boundary value problems**

▷ **Green’s functions**
  Dirichlet and Neuman boundary value problems and initial boundary value problems. Construction of Green’s functions for specific domains.

▷ **Transform methods**
  Fourier, Finite Fourier, Hankel and Laplace transforms on bounded, semi-infinite and infinite domains. Application of these transforms to problems in Cartesian, polar, cylindrical and spherical coordinates.

Tables of transforms will be provided, as needed.

**Suggested references**


Relevant material for Unit II is in Chapters 1 to 12 and 15 in [4] and Chapters 1 to 5 in [5], except for sections 2.4-2.6, 2.9, 2.10. Note that both books are available online through the University of Manitoba Libraries.
Topology Comprehensive Examination Syllabus

1: Basic Point-Set Topology

3. Connectedness: Connectedness, path connectedness, (connected) components, and path components.
5. Separation axioms: \(T_0\), \(T_1\), \(T_2\) (Hausdorff), regularity, \(T_3\), complete regularity, \(T_{3\frac{1}{2}}\) (Tychonoff), normality, \(T_4\), and the relationships among them. Urysohn’s Lemma, and Tietze’s Extension Theorem.
6. Countability axioms: Separability, first countability, second countability, the Lindelöf property, and the relationships among them.
7. Metric spaces: Topological properties, as itemized in 1–6, and their characterizations in metric spaces.

Suggested reading

   §1-§35, §37, §43, §45, §48.
   §1-§9, §13-§19, §22-§27.

2: Algebraic Topology

3. The fundamental group. Simply connected spaces. Fundamental group of a circle.
5. Seifert-van Kampen theorem and consequences.
8. Applications:
   ⊳ Brouwer Fixed Point Theorem
   ⊳ Borsuk-Ulam Theorem

**Suggested reading**

   Chapters 1 to 5.

   University of Manitoba Libraries’ online catalogue].
   Chapters 1 to 12.