University of Manitoba Computational Mathematics Comprehensive Examination April 2018

Examining Committee: R. M. Slevinsky (coordinator), K. Kopotun, A. Prymak

Name: ____________________, Student I.D.#: ____________________

9:00 a.m. – 3:00 p.m., April 30, 2018

The exam consists of two parts worth 40 marks each. The passing criterion is at least 60 marks overall. Students have 6 hours to complete the exam.

Part I: Introductory Numerical Analysis

This unit consists of five questions worth a total of 40 marks. Answer all questions.

1. [5 marks]
2. [7 marks]
3. [8 marks]
4. [10 marks]
5. [10 marks]

Part II: Numerical Analysis of PDEs

This unit consists of five questions worth 10 marks each. Answer four questions. You may attempt as many questions as you like in this unit; however, if you attempt more than four questions, you must clearly indicate which answers you want us to mark. In the absence of any explicit indication, we will mark the first four questions.

1. [10 marks]
2. [10 marks]
3. [10 marks]
4. [10 marks]
5. [10 marks]

Part I: Introductory Numerical Analysis

This unit consists of five questions worth a total of 40 marks. Answer all questions.

1. [5 marks] Consider $n+1$ data points $(x_0, f_0), \ldots, (x_n, f_n)$ with distinct abscissæ, $x_i \neq x_j \; \forall i \neq j$. Prove that there exists a unique degree-n polynomial that interpolates the data.

2. [7 marks] The Lorenz curve is the one-to-one monotonically increasing function $y = L(x) : [0,1] \to [0,1]$ that informs us of the fraction of the total income earned by the poorest fraction of all households in a particular jurisdiction. The table below lists data on the total incomes of the quartiles of a country's households.

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$L(x)$	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{2}$	1

   We can hardcode the assumptions $L(0) = 0$ and $L(1) = 1$ by a reasonable change of dependent coordinates:

$$L(x) = x\left[(x-1)f(x) + 1\right],$$

   for some function $f(x)$.

   (a) [1 mark] Rescale the interior Lorenz data by the transformation:

$$f(x) = \frac{L(x)/x - 1}{x - 1}.$$

   (b) [2 marks] Using Lagrange basis polynomials, find the Lagrange interpolating polynomial to the rescaled data.

   (c) [2 marks] Show that it is equal to $p_2(x) = \frac{64}{15}x^2 - \frac{16}{5}x + \frac{4}{3}$.

   (d) [2 marks] Approximate the Gini index, $G = 2\left\|x - L(x)\right\|_1$, by using the interpolating polynomial from part (c). There is no need to simplify.

3. [8 marks] Find the first four orthogonal polynomials in $L^2([-1,1], |x|\,\mathrm{d}x)$. What are the nodes and weights of the associated three-point Gaussian quadrature rule?

4. [10 marks] Consider the initial-value problem:

$$y' = f(t, y), \quad y(t_0) = y_0,$$

   where $y_0 \in \mathbb{R}$ and $f : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is infinitely differentiable.

   For every $\alpha \in (0, 1]$, consider the two-stage explicit Runge–Kutta method:

$$y_{n+1} = y_n + h\left[\left(1 - \frac{1}{2\alpha}\right) f(t_n, y_n) + \frac{1}{2\alpha} f(t_n + \alpha h, y_n + \alpha h f(t_n, y_n))\right].$$

   Find the leading coefficient of the local truncation error.

5. [10 marks] Consider the matrix:

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

   (a) [2 marks] Write down the Jacobi iteration for the linear system $Ax = b$.

   (b) [8 marks] Prove that the iterations converge for any initial iterate.

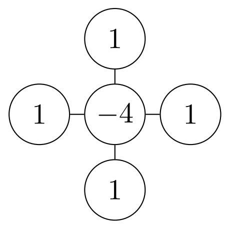
Part II: Numerical Analysis of PDEs

This unit consists of five questions worth 10 marks each. Answer four questions. You may attempt as many questions as you like in this unit; however, if you attempt more than four questions, you must clearly indicate which answers you want us to mark. In the absence of any explicit indication, we will mark the first four questions.

1. [10 marks] Consider the Poisson equation in the square $\Omega = (0,1)^2$:

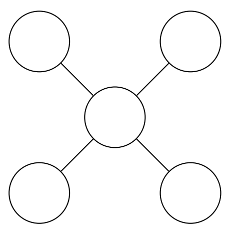
$$u_{xx} + u_{yy} = f, \qquad u(x,y) = 0, \quad \text{for} \quad (x,y) \in \partial\Omega.$$

   (a) [4 marks] The classical five-point formula leads to the computational stencil:



$$u_{j,k} = h^2 f_{j,k}.$$

   where $h = \Delta x = \Delta y$ and $u_{j,k} = u(jh, kh)$ and $f_{j,k} = f(jh, kh)$ for $j, k = 1, \ldots, n$. Fill in the computational stencil:



$$u_{j,k} = h^2 f_{j,k},$$

   to ensure it has a local truncation error of the same order as that of the classical five-point formula.

   (b) [3 marks] Is the discretization matrix you created symmetric? negative-definite?

   (c) [3 marks] Describe a direct method for the solution of the discretized modified five-point formula. Choose an ordering to improve the complexity and state the improved complexity (in terms of n).

2. [10 marks] Consider the heat equation:

$$\begin{aligned} &u_t = u_{xx} + f(x,t), \quad x \in (0,1), \quad t > 0, \\ &u(x,t) = 0, \quad x = 0, \quad x = 1, \quad t > 0, \\ &u(x,0) = g(x), \quad x \in (0,1). \end{aligned}$$

   (a) [3 marks] Write down the Crank–Nicolson scheme;

   (b) [7 marks] Analyze the stability using either (i) eigenvalue, or (ii) Fourier (von Neumann) analysis.

3. [10 marks] Analyze the stability of the forward in time backward in space first order finite difference scheme for the advection equation:

$$\begin{aligned} &u_t + u_x = 0, \quad x \in (0,1), \quad t > 0, \\ &u(x,0) = f(x), \quad u(0,t) = g(t). \end{aligned}$$

   using:

   (a) [2 marks] eigenvalue analysis; and,

   (b) [6 marks] Fourier (von Neumann) analysis.

[2 marks] When you find two different conditions on the Courant number, which one is correct and why?

4. [10 marks] Consider the PDE $-\Delta u + u = f \in L^2(\Omega)$, where $u \in H_0^1(\Omega) \cap H^2(\Omega)$.

   (a) [2 marks] Derive the weak form of the PDE.

   (b) [3 marks] Prove that the bilinear form you find in (a) is continuous and $H_0^1(\Omega)$-elliptic (bounded and coercive in $H_0^1(\Omega)$).

   (c) [3 marks] Conclude that there exists a unique weak solution.

   (d) [2 marks] What is the regularity requirement on V_h, the usual finite element subspace of $H_0^1(\Omega)$? What are the usual finite elements for an open and bounded $\Omega \subset \mathbb{R}^d$?

5. [10 marks] Consider the linear system:

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   (a) [8 marks] Using the initial guess $x^{(0)} = [0, -2]^\top$, solve the linear system by the method of conjugate gradients. *Recall: after initializing $p^{(0)} := r^{(0)} := b - Ax^{(0)}$, one step ($k \to k+1$) in the conjugate gradient method is:*

$$\begin{aligned} \alpha_k &= \frac{\langle r^{(k)}, r^{(k)} \rangle}{\langle p^{(k)}, Ap^{(k)} \rangle}, \\ x^{(k+1)} &= x^{(k)} + \alpha_k p^{(k)}, \\ r^{(k+1)} &= r^{(k)} - \alpha_k Ap^{(k)}, \\ \beta_k &= \frac{\langle r^{(k+1)}, r^{(k+1)} \rangle}{\langle r^{(k)}, r^{(k)} \rangle}, \\ p^{(k+1)} &= r^{(k+1)} + \beta_k p^{(k)}. \end{aligned}$$

   (b) [2 marks] State a property of the method that proves that you have attained the solution.