Instructions:

• You have altogether 6 hours to write the exam.

• Please write all your responses in the provided booklets.

• This exam has 3 parts. Part A consists of 10 questions worth two marks each. Please answer all questions in Part A.

You have a choice of questions in each of Part B and Part C. In Part B, please answer 6 of 10 questions; each response is worth 5 marks.

In Part C, please answer 4 of 6 questions; each response is worth 10 marks.

If you answer more questions than required, please indicate clearly which questions you would like to be marked. Otherwise, the first 6 (resp. 4) responses will be marked.

There are a total of 90 marks in this exam.

• In order to pass the examination, you must obtain an overall score of at least 75% \( (\approx 0.675) \) in total.

• Unless otherwise stated, rings are assumed to contain a multiplicative identity 1. In your responses, it may be helpful to keep the following in mind: unless otherwise indicated,

  – vector spaces need not be finite-dimensional;
  – groups are not necessarily finite, and not necessarily abelian;
  – fields may be finite or infinite, of arbitrary characteristic;
  – rings may not necessarily be commutative;
  – \( S_n \) denotes the group of permutations of the set \( \{1, \ldots, n\} \).
Part A

Answer each of the following 10 questions; each question is worth 2 marks. Your responses should be brief, two or three sentences should suffice.

A.1. Describe all the ideals of the ring $\mathbb{Q} \oplus \mathbb{Q}$.

A.2. Define what it means for an extension of fields $L/K$ to be separable.

A.3. Let $K$ be a field. Find an ideal of $K[x, y]$ that is prime but not maximal. Be sure to justify your answer.

A.4. Let $A$ be a $n \times n$ skew-symmetric matrix (i.e. $A^T = -A$) with real coefficients, where $n$ is odd. Show that $A$ is not invertible.

A.5. Let $R$ be a ring and $I \subset R$ a non-trivial proper ideal. Prove or disprove: $I$ is a free $R$-module.

A.6. Let $R$ be a ring. Prove or disprove: the sum of two nilpotent elements in $R$ is nilpotent.

A.7. Show that $\mathbb{Z}$ is not Artinian.

A.8. Show that the centre of a group is a normal subgroup.

A.9. Let $V$ be a vector space. Show that any maximal linearly independent set of vectors in $V$ is a basis.

A.10. Show that every group of order $n$ is isomorphic to a subgroup of $S_n$.

Part B

Answer 6 out of the following 10 questions. Each question is worth 5 marks. If you answer more than 6, please indicate clearly which questions you would like to be marked; otherwise, the first 6 responses will be marked.

B.1. Suppose $T$ is a self-adjoint operator on a complex Hilbert space $V$, and $v_1$ and $v_2$ are eigenvectors with eigenvalues $\lambda_1 \neq \lambda_2$. Prove that $v_1$ and $v_2$ are orthogonal.

B.2. Prove that $S_n$, the group of permutations on $n$ elements, is generated by the set of all transpositions $\{(ij) \mid 1 \leq i < j \leq n\}$.

B.3. Recall that an abelian group $G$ is divisible if for each $g \in G$ and each positive integer $n$, there exists $k \in G$ such that $g^n = k$. Show that any quotient of a divisible group is divisible. Show that every non-trivial divisible abelian group is infinite.

B.4. Let $F$ be a finite field of characteristic $p$. Show that the Frobenius map $\varphi: F \to F$, defined by $\varphi(x) = x^p$, is a bijection.
B.5. Give an example of each of the following, with a brief justification:

(a) A PID (principal ideal domain) that is not Euclidean.
(b) A unique factorization domain that is not PID.

B.6. Show that a finite dimensional vector space is isomorphic to its dual space.

B.7. Classify all groups, up to isomorphism, of order 10; present your classification in terms of generators and relations.

B.8. Show that if \([\mathbb{Q}(\alpha) : \mathbb{Q}]\) is odd, then \(\mathbb{Q}(\alpha^2) = \mathbb{Q}(\alpha)\).

B.9. Show that \((\mathbb{Z}/10\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/15\mathbb{Z})\) is cyclic as an abelian group, and find its order.

B.10. Show that if \(H\) is a subgroup of \(G\) of index 2, then \(H\) is normal.

**Part C**

Answer 4 out of the following 6 questions. Each question is worth 10 marks. If you answer more than 4, please indicate clearly which questions you would like to be marked; otherwise, the first 4 responses will be marked.

C.1. Let \(p\) be a prime, and \(\mathbb{F}_p\) the finite field with \(p\) elements. Show that for each integer \(n > 1\) there exists an extension \(F_n/\mathbb{F}_p\) of degree \(n\), and that \(F_n\) is unique up to isomorphism.

Hint: Consider the polynomial \(x^{p^n} - x \in \mathbb{F}_p[x]\).

C.2. (a) Give the definition of a solvable group.
(b) Suppose \(G\) is a group and \(N \subset G\) a normal subgroup. Show that \(G\) is solvable if and only if both \(N\) and \(G/N\) are solvable.

C.3. Consider the field \(K = \mathbb{Q}(\sqrt{3}, i)\). Show that \(K\) is Galois over \(\mathbb{Q}\) and find its Galois group.

C.4. Let \(p\) be a prime. Show that any group of order \(p^2\) is abelian.

C.5. Let \(R\) be a commutative ring with unity and \(M\) an \(R\)-module. Show that \(M\) is simple if and only if it is isomorphic to a module of the form \(R/I\) for some maximal ideal \(I \subset R\).

C.6. Let \(R\) be a ring, and \(M\) an \(R\)-module. Show that the (contravariant) functor \(\text{Hom}_R(\cdot, M)\) is right exact, i.e. given an exact sequence

\[ A \rightarrow B \rightarrow C \rightarrow 0 \]

of \(R\)-modules, show that the induced sequence

\[ 0 \rightarrow \text{Hom}_R(C, M) \rightarrow \text{Hom}_R(B, M) \rightarrow \text{Hom}_R(A, M), \]

is exact.