University of Manitoba
Department of Mathematics

Graduate Comprehensive Examination in Algebra

10:00 AM– 4:00 PM 31 January, 2018.

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Instructions (Please read carefully):

- You have altogether 6 hours to complete the examination.
- Part A consists of 10 questions worth two marks each. Answer all questions in Part A on the question paper itself. Each of these questions can and should be answered in no more than three sentences.
- You have a choice of questions in each of Parts B and C. The questions in Part B are worth 5 marks each. Answer any 6 questions out of 10 in this part. The questions in Part C are worth 10 marks each. Answer any 4 questions out of 6 in this part.
- You may attempt as many questions as you like in Parts B and C; however, if you attempt more than the required number of questions, you must clearly indicate which answers you want evaluated. In the absence of any explicit indication, the first 6 questions for Part B, and the first 4 questions for Part C (in the order of their appearance in your answer booklets) will be evaluated.
- In order to pass this examination, you must obtain a score of at least 75% in total.

Be sure to keep in mind the following:

- Unless stated otherwise, vector spaces need not be finite dimensional.
- Unless stated otherwise, groups may be finite or infinite, abelian or non-abelian.
- Unless stated otherwise, rings may be commutative or non-commutative.
- Unless stated otherwise, rings R **are** assumed to have a multiplicative identity $1 \in R$.
- Unless stated otherwise, fields may be finite or infinite, of arbitrary characterstic.
- S_n denotes the group of permutations on the set $\{1, \ldots, n\}$.

PART A

Please answer each of the following 10 questions in the space provided. Each correct answer is worth two marks. Each question should be answered briefly; i.e., in no more than three sentences.

A1. Let V be a finite dimensional real vector space, and let $J : V \to V$ be an operator that satisfies $J^2 = -\mathrm{id}_V$. Show that J is not diagonalizable.

A2. Let $\phi : R \to S$ be a homomorphism between commutative rings. Prove or give a counterexample: If $x \in R$ is a zero divisor, then so is $\phi(x) \in S$.

A3. Determine the order of A_n, the group of even permutations.

A4. What is the torsion subgroup of $\mathbb{R}/\mathbb{Z}$?

A5. Let P_n be the vector space of polynomials with coefficients in $\mathbb{R}$ of degree at most n. Let $D : P_n \to P_n$ be the operator $D(p(x)) = \frac{d}{dx}p(x)$. Show that $\lambda = 0$ is the only eigenvalue of D.

A6. What are the possible Jordan normal forms for a 3×3 matrix over $\mathbb{C}$?

A7. Suppose that the degree of the field extension $F \subset K$ is a prime p. Show that any subfield E of K containing F is equal to either K or F.

A8. Let R be a commutative ring. Define "R is Artinian."

A9. Let R be a ring, M a right R-module and N a left R-module. State the universal property of the tensor product $M \otimes_R N$.

A10. Prove that the extension $\mathbb{Q} \subset \mathbb{Q}(2^{1/4})$ is not a Galois extension.

PART B

Please answer any 6 of the following 10 questions in your answer booklet. Each question is worth 5 marks. If you attempt more than 6 questions, then please indicate clearly which ones you want evaluated.

B1. Show that an $n \times n$ matrix A is invertible if and only if it satisfies the *cancellation* law: $XA = YA$ implies $X = Y$ for all $n \times n$ matrices X, Y.

B2. Let G be a group.

(a) Define "G is nilpotent."

(b) If H is a subgroup of a nilpotent group, show that H is also nilpotent.

B3. Let $Z = Z(G)$ be the centre of a group G. If G/Z is cyclic, show that G is abelian.

B4. Let V be a finite dimensional vector space and $\phi : V \to V$ a linear transformation satisfying $\phi^2 = \phi$.

(a) Prove that $\ker(\phi) \cap \text{image}(\phi) = \{0\}$.

(b) Prove that $V = \ker(\phi) \oplus \text{image}(\phi)$.

B5. Let R be a principal ideal domain and let S be a ring with identity $1 \neq 0$. Let $\phi : R \to S$ be a surjective ring homomorphism. Prove that every ideal in S is principal.

B6. Let p be a prime. Prove that every group of order p^2 is abelian. You may use the result of question B3, even if you have not completed that question.

B7. Let $D_8 = \langle r, s \,|\, r^4 = s^2 = e,\, rs = sr^{-1} \rangle$ be the dihedral group and consider the homomorphism $\phi : D_8 \to \mathrm{GL}_2(\mathbb{C})$ given by

$$\phi(r) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \phi(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

A subspace $W \subset \mathbb{C}^2$ is D_8-*invariant* if for all $g \in D_8$, $\phi(g)(w) \in W$ for all $w \in W$. Show that there is no non-zero, proper subspace $W \subset \mathbb{C}^2$ that is D_8-invariant.

B8. Show that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic as fields.

B9. Let P be a Sylow p-subgroup of H and let H be a subgroup of K. If P is normal in H and H is normal in K, prove that P is normal in K.

B10. Show that $\mathbb{Z}[\sqrt{-5}] := \{a + b\sqrt{-5} \mid a, b, \in \mathbb{Z}\}$ is an integral domain that is not a principal ideal domain.

PART C

Please answer any 4 of the following 6 questions in your answer booklet. Each question is worth 10 marks. If you attempt more than 4 questions, then please indicate clearly which ones you want evaluated.

C1. Let G be a group of order 385. Prove that the centre of G contains a Sylow 7-subgroup of G.

C2. Determine the degree of the extension $\mathbb{Q}\left(\sqrt{\frac{1+\sqrt{-3}}{2}}\right)$ over $\mathbb{Q}$. Is this a Galois extension? If yes, what is its Galois group?

C3. Let R be a nonzero commutative ring, M an R-module and $S \subseteq R$ a multiplicatively closed subset. Prove that there exists a unique isomorphism

$$f : S^{-1}R \otimes_R M \to S^{-1}M$$

for which

$$f((r/s) \otimes m) = rm/s$$

for all $r \in R$, $m \in M$ and $s \in S$.

C4. Let W_1 and W_2 be finite-dimensional subspaces of a vector space V over a field $\mathbb{F}$. Prove that

$$\dim_F(W_1 + W_2) = \dim_F(W_1) + \dim_F(W_2) - \dim_F(W_1 \cap W_2).$$

C5. Let R be a commutative ring. Prove that R is Noetherian if and only if every ideal is finitely generated.

C6. Let S_n denote the group of permutations of n elements.

(a) Show that every element of S_n is a product of disjoint cycles.

(b) If $\sigma \in S_n$ is the product of disjoint cycles of length $n_1, n_2, \ldots, n_r$ with $n_1 \leq \ldots \leq n_r$ (here we include 1-cycles) then we define the *cycle type* of σ to be the r-tuple $(n_1, \ldots, n_r)$. Prove that two elements of S_n are in the same conjugacy class if and only if they have the same cycle type.

—END OF EXAM—