Combinatorics comprehensive, Fall 2015

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9:00 AM Wednesday 30 September 2015

This exam has two parts, A and B. Answer all 10 questions from part A. Answer at most 5 of the 8 questions from part B, clearly indicating which questions you want graded. Each problem in part A is worth 5 points. Each question in part B is worth 10 points. The total possible score for this test is 100 points. A pass is 75/100. The total time for this exam is 6 hours.

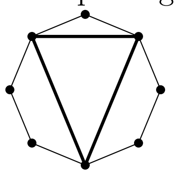
No outside references or electronic aids are allowed.

Unless otherwise noted, all variables are non-negative integers and all graphs are simple. The vertex set of a graph G is denoted by $V(G)$ and the edge set is denoted by $E(G)$.

1 Part A

Do all problems. Each problem is worth 5 points.

[5] **A1.** Let P be a convex polygon with $n > 4$ sides. How many triangles can be formed whose vertices are vertices of the polygon, but whose sides are not sides of the polygon? (An example is given in the picture below.) Prove the formula you obtain.



[5] **A2.** Let n_k denote the number of binary 0-1 sequences of length k with no two zeros adjacent. Find, with proof, a simple well-known expression for n_k.

[5] **A3.** Let a and b be positive integers. Prove that the number of sequences of positive integers $(z_1, z_2, \ldots, z_b)$ satisfying $1 \leq z_1 \leq z_2 \leq \cdots \leq z_b \leq a+1$ is $\binom{a+b}{a}$.

[5] **A4.** Which Fibonacci numbers are divisible by 5? Prove your answer.

[5] **A5.** Let G be a bipartite graph on $n \geq 3$ vertices with at least one edge. Show that $n/2 \leq \chi(\overline{G}) \leq n-1$.

[5] **A6.** Prove that if a simple graph G on $n \geq 3$ vertices is Eulerian, then G has at least three vertices with the same degree.

[5] **A7.** Find three mutually orthogonal latin squares (MOLS) of order 4, or show that no such three exist.

[5] **A8.** For positive integers $m, n \geq 2$, define the Ramsey number $R(m,n)$. Find, with proof, $R(3,4)$.

[5] **A9.** A deck of 25 cards has five suits (spades, clubs, hearts, diamonds, bananas) and each suit has cards of five ranks: Ace, 2, 3, 4, 5. Is it possible to arrange the cards from this deck in a 5×5 array so that each rank and each suit appears exactly once per row and once per column? Either disprove or produce such an array.

[5] **A10.** Find the number of paths in the integer lattice that go up or to the right, that start from (0,0) and end at (10,10) that do not go ABOVE the line $y = x + 1$.

2 Part B

Do any five problems. Clearly indicate which problems are to be graded. Each problem is worth 10 points.

[10] **B1.** Show that any triangle-free graph G on n vertices satisfies $\chi(G) \leq 2\sqrt{n}$.
[Hint: one such proof uses strong induction on the order of the graph.]

[10] **B2.** Suppose that $n_0 = 0$, $n_1 = 1$ and for each $k \geq 2$, $n_k = 3n_{k-1} - 2n_{k-2}$. Solve this recursion using either the method of characteristic roots or generating functions, giving an explicit formula (in closed form) for n_k. Prove your answer by mathematical induction.

[10] **B3.** A graph is *chordal* iff every cycle of length at least 4 has a chord. If H, G_1 and G_2 are graphs and both G_1 and G_2 contain a copy of H as an induced subgraph, define the *amalgamation* of G_1 and G_2 along H ito be the graph G formed by identifying vertices in each copy of H.

Show that if a chordal graph G is not a complete graph, then G is the amalgamation of two chordal graphs along a complete graph.

[10] **B4.** A Steiner triple system (STS) is a $(v, b, 3, r, 1)$-BIBD. Prove that if a STS exists on v varieties, then $v \equiv 1$ or $3 \pmod 6$.

[10] **B5.** Let n be even and let G be the graph formed by removing a 1-factor from K_n. Find the number of closed walks of length 5 from any vertex back to itself.

[10] **B6.** Give a generating function for the number of partitions of n into ODD summands. Give a generating function for the number of ways of partitioning n into DISTINCT summands. Show that these two numbers are the same by comparing generating functions.

[10] **B7.** Let G be a graph on n vertices and m edges. Show that if $m > \frac{n^2}{4} + 1$, then G contains two triangles sharing only one common vertex.

[10] **B8.** Suppose that (X, R) is a distributive lattice. Show that the lattice is *modular*, that is, if yRx, then for all $z \in X$,

$$y \vee (x \wedge z) = x \wedge (y \vee z).$$