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The Top Ten: Essential Problem Solving Techniques for High School Contests

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Problem solving often seems to be the most intimidating part of mathematics, largely due to the profusion of different ways of approaching a given question. There is no clearly defined path to a solution, and many of the promising ideas that seem to offer the hope of an answer lead only to failure and frustration. However, it is exactly this lack of constraints on the way a problem can be solved that make it such an engaging exercise. It tests not only mechanical skills of mathematics, but also creativity, ingenuity, and resourcefulness in taking what you've learned and generalizing it to unexpected situations. The manner in which you apply your knowledge will depend primarily on past experiences with similar sorts of problems: by identifying commonalities between a given problem and ones you've solved before, you will begin to accumulate a repertoire of problem solving techniques that will help you on exams and contests.
The purpose of this article is to introduce first a generic method of problem solving that provides a foundation for the development of problem solving skills and then a list of the techniques that I found particularly useful as a high school student when solving contest questions.

1 Polya's method

If not the first, then the most widely accepted universal approach to problem solving was introduced by George Polya in his book *How to Solve It* [1]. Although its generality hampers its use when solving a specific problem, it provides a good framework for tackling any question and profiting from the experience, since it is thoroughly common-sensical. There are four parts to it:

1.1 Understand the problem

This sounds trivial, but without grasping what the question is asking, you're doomed to failure. At the high school level the meaning of the problem is usually quite clear. Nonetheless, be careful about notation: for example, when dealing with an inequality, do I need to prove that one quantity is strictly greater ($>$) or greater than or equal to ($\geq$) another quantity?
As well, though most questions will explain more uncommon notation, there are certain characters whose meaning it is profitable to know offhand, both for ease of comprehension and speed when writing up answers, like $\propto$ (proportional to), $\Rightarrow$ (implies that), $\exists$ (there exists), $\forall$ (for all), $|$ (such that), $\cup$ (the union of two sets), and $\cap$ (the intersection of two sets). Using these "shortcut" symbols when writing up solutions in a contest environment can save time better spent thinking about math, rather than English!

Continued on page 6

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Submissions

Manitoba Math Links welcomes material on any topic related to mathematics, including articles, applications, announcements, humour, anecdotes, problems, and history. Materials are subject to editorial revision. Submissions may be made by regular mail or electronically to either the general address, or directly to any of the editors.

We also welcome editorial comments or suggestions from students, teachers, parents, or anyone interested in mathematics and mathematics education. The format of this newsletter is constantly evolving, so any input is appreciated.

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SPEAKERS AVAILABLE

If you are interested in having a faculty member come to your school and speak to students, please contact the Department of Mathematics or any member of the editorial board.

Contents

Classic puzzles

D. Gunderson

In *Scientific American*, Martin Gardner ran the column "Mathematical Games" for over 25 years. Each column usually had short problems, and these have been collected recently in the splendid new book *The colossal book of short puzzles and problems*, edited by Dana Richards, W. W. Norton & Company, 2006. Here are two:

Problem 5.13: A chessboard has squares that are two inches on the side. What is the radius of the largest circle that can be drawn on the board in such a way that the circle's circumference is entirely on black squares?

Problem 14.8 (from June 1961): *The square root of wonderful* was the name of a play on Broadway. If each letter in WONDERFUL stands for a different digit (zero excluded) and if OODDF, using the same code, represents the square root, then what *is* the square root of wonderful?

Answers appear in next issue. If you need the answer sooner, please email me (or buy the book).

Cool Websites

R. *Padmanabhan*
Department of Mathematics

A BINARY MAGIC

You are a mathematical magician. Select three volunteers from the audience (your class, your family). Ask each one of them to think of a number between 1 and 31. You then display five cards containing numbers as given below. Then ask each volunteer to determine which cards contain the number he or she selected.

Card A

1	9	17	25
3	11	19	27
5	13	21	29
7	15	23	31

Card B

2	10	18	26
3	11	19	27
6	14	22	30
7	15	23	31

Card C

4	12	20	28
5	13	21	29
6	14	22	30
7	15	23	31

Card D

8	12	24	28
9	13	25	29
10	14	26	30
11	15	27	31

Card E

16	20	24	28
17	21	25	29
18	22	26	30
19	23	27	31

"A, C and E" declares one volunteer. "Then your number is 21" is your quick reply.

"My number is only on card E" says the second volunteer. "Abra cadabra, your number is 16."

"My number is on cards B, C, D and E," proclaims the third volunteer. "Oh, your number must be 30," you declare triumphantly.

And every time you will be correct. Is it magic? No, it is just mathematics, pure and simple. While it looks impressive, it relies on simple principle of binary place value notation of numbers. The key idea is the proper distribution of numbers in the various cards. If you notice, the top left hand number in each card is a power of two. The various powers of 2 are 1, 2, 4, 8, 16, 32 and so on.

Let us take the third answer above to explain the way the cards are designed: the powers of 2 occurring in the cards B, C, D and E are 2, 4, 8 and 16. Now 30 is the unique number that occurs in all the four cards and it does not occur in A. Let us see why using the binary representation for 30:

$$30 = 16 + 14 = 16 + 8 + 6 = 16 + 8 + 4 + 2.$$

Thus, to determine the number your opponent has selected, all you need do is add up the top left hand number on each of these cards. This will be the number they chose.

You can use the same idea to make cards with more numbers, say six cards containing numbers from 1 to 63. Now we have six powers of 2: 1, 2, 4, 8, 16 and 32. Let us determine where the number 55 will get entered in this set of six cards:

$$55 = 32 + 23 = 32 + 16 + 7 = 32 + 16 + 4 + 3 = 32 + 16 + 4 + 2 + 1.$$

Hence 55 will go in the cards with 1, 2, 4, 16 and 32 as their top left hand numbers. In this case you opponent will say, "my number is on cards A, B, C, E and F." Again, all you need do is add up the top left hand number on each of these five cards and declare the number as "55". This is really cool.

For more info on binary numbers and their magic properties, please visit:
`http://www.math.grin.edu/~rebelsky/Courses/152/97F/Readings/student-binary.html`
`http://www.gomath.com/Questions/question.php?question=48679`
`http://www.math.princeton.edu/matalive/Crypto/CryptoLab1/Conversion.html`

To play this game online, go to the website:
`http://gwydir.demon.co.uk/jo/binary/how.htm`
`http://gwydir.demon.co.uk/jo/binary/cards.htm`

For a guessing game with binary numbers and connection to error correcting codes:
`http://unplugged.canterbury.ac.nz/activities/error.pdf`
`http://cadigweb.ew.usna.edu/~wdj/montague/montague_mathhonors1998-1999.html`
`shttp://www.math.uncc.edu/~hbreiter/SVSM/taurispaper.html`

A GEOMETRY PROBLEM WITH THREE SOLUTIONS

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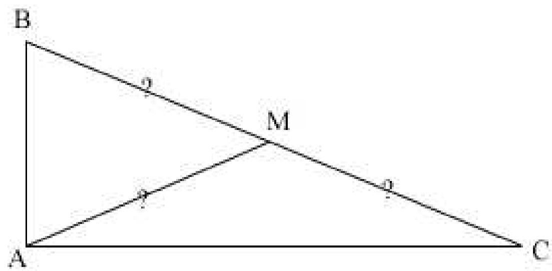
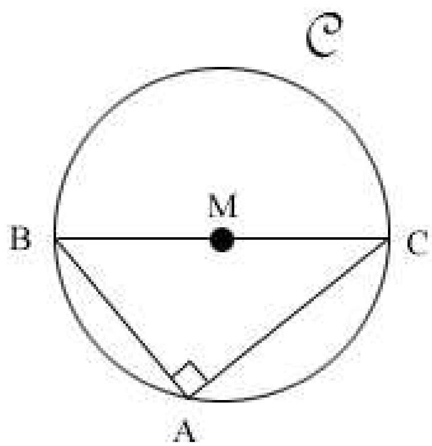
Often a geometry problem is most easily solved by adding some new line segments to the problem that were not present in the original presentation of the problem. In the problem that follows, we exhibit three different ways of doing this.

Before you read the solutions, try the problem yourself. See if your solution resembles any of the solutions presented below.

Problem: Show that in a right angled triangle ABC (with right angle at A) the length of the median AM (*i.e.*, the line segment from A to the midpoint M of the hypotenuse BC) is half the length of the hypotenuse.



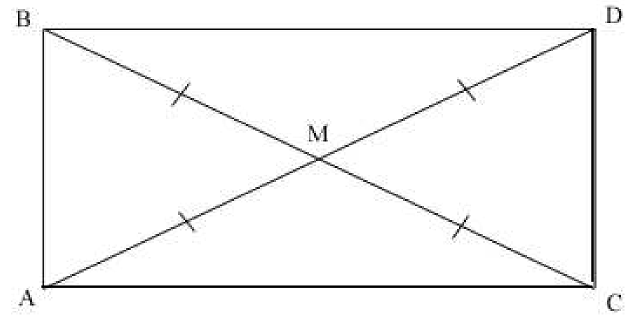
Solution 1: Any three non-collinear (not in a straight line) points lie on a unique circle $\mathcal{C}$. (You can find the centre of the circle by finding the point where the perpendicular bisectors of any two sides meet.)



Now the fact that $\angle BAC$ is a right angle means that the line segment BC must be a diameter of $\mathcal{C}$ (ask your teacher about this). Therefore, the midpoint M of BC must be the centre of the circle, and so MA, MB, and MC are all radii of $\mathcal{C}$, and hence have the same length. [This solution uses rather heavy duty properties of circles.]

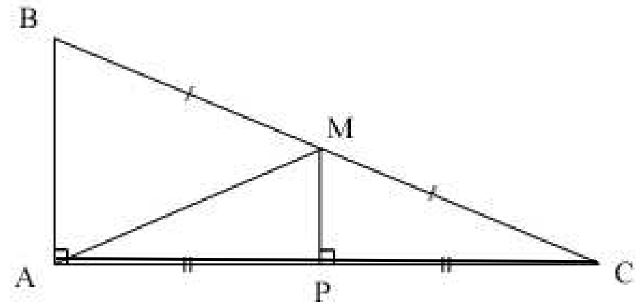
[1]The author thanks Prof. G. Woods for kindly rewriting this article and Rob Borgersen for the diagrams.

Solution 2: Points A, B, and C form three of the four vertices of a rectangle. Let D be the fourth vertex (opposite A).



Then BC and AD are the diagonals of the rectangle, and they intersect at a common midpoint, *i.e.*, at M. But BM, MC, and AM are each half the length of the diagonal of rectangle $ABCD$, and hence (by properties of rectangles), are equal.

Solution 3: Drop a perpendicular from M to the line segment AC, intersecting AC at P.



Since $AB \perp AC$ and $MP \perp AC$, then AB and PM are parallel. Now since $\triangle ABC$ and $\triangle MPC$ share the angle $\angle BCA$, and are right triangles, they are similar triangles. Thus by Thales' theorem,

$$\frac{CM}{BM} = \frac{CP}{AP}.$$

Using $CM = BM$, this gives $CP = AP$.

Then $\triangle APM$ and $\triangle CPM$ have two pairs of sides equal (*i.e.*, AP and CP, and the common side MP) and the enclosed angles $\angle APM$ and $\angle CPM$ equal (both are right angles). Thus, these triangles are congruent, so the corresponding sides AM and CM must be equal.

Did *you* find a different solution?

> The advancement and perfection of Mathematics are intimately connected with the prosperity of the state.
>
> —*Napoléon Bonaparte (1769–1821)*

Three different outcomes to the same problem in population dynamics

Julien Arino
Department of Mathematics, U of M

Consider a population of rabbits and suppose that the rate at which new rabbits are born depends on the current number of rabbits. Also, suppose that rabbits eat only carrots, for which they compete with each other: if there are too many rabbits, some rabbits starve to death; this is known as *intraspecific competition*, because it takes place within a species (*interspecific* competition occurs when rabbits compete for carrots with, say, the Green Giant).

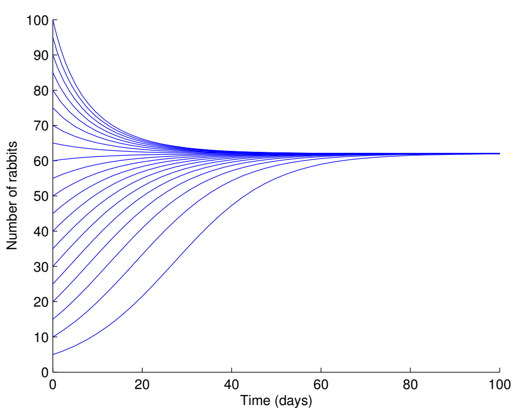
Let t be a real number representing days and $N(t)$ be the number of rabbits on day t (t can take any positive value). The *logistic* equation,

$$N'(t) = rN(t)\left(1 - \frac{N(t)}{K}\right),$$

describes the evolution of the rabbit population. Remark that it involves a *derivative*, $N'(t)$; it is called an *ordinary differential equation* (ODE), and the unknown, $N(t)$, is a function rather than a number.

Solving ODEs requires advanced mathematics, but the logistic equation is not too difficult (although well beyond the scope of this presentation). The constant r is the *intrinsic growth rate*, that is, how many new rabbits would be born every day if there were an unlimited supply of carrots, and K is the *carrying capacity* of the environment, the number of rabbits that can survive with the limited number of carrots that is effectively there.

In the following figure, several solutions to the logistic ODE are represented, corresponding to different initial numbers of rabbits; the horizontal axis shows time and the vertical axis shows the number of rabbits, so following a curve from left to right indicates the evolution of the population through time.

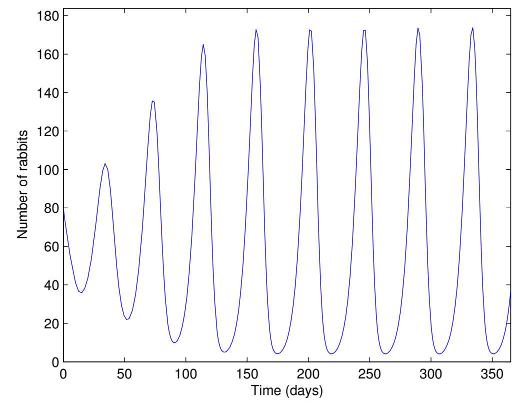


All the solutions tend to the same value, which turns out to be K, the carrying capacity of the environment.

Now suppose that it takes τ days between the instant rabbits compete for a carrot and the outcome of this competition, that is, the death of a rabbit. In this case, we use a *delay differential equation* (DDE),

$$N'(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right).$$

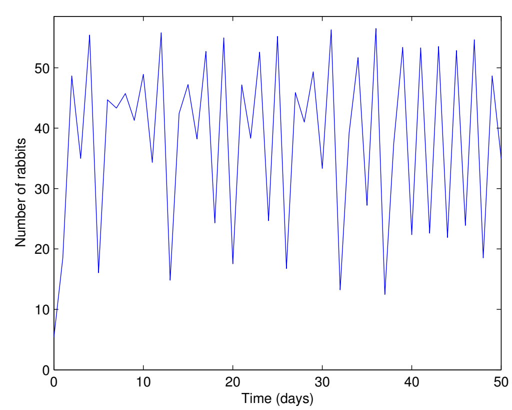
The constant τ is called the *time delay*. DDEs are much more complicated than ODEs, and have been studied only since the 1950s. In the next figure, a solution to the logistic DDE is represented. The big difference with the logistic ODE is that the population of rabbits oscillates.



The last version of the logistic equation uses *discrete* time, where time is an integer quantity instead of varying continuously as in the previous two equations. The solution is a *sequence* where the number of rabbits on day $t+1$ is given as a function of the number of rabbits on day t:

$$N(t+1) = rN(t)\left(1 - \frac{N(t)}{K}\right).$$

Because the solution is not continuous, this is also more difficult than the logistic ODE. The next figure shows a solution to this equation (the graph should consist of dots, but for visualization, they are connected):



Such a solution is *chaotic*: it oscillates, but in a very irregular way, and two solutions that start very close to each other will be very different after a small time. The discrete logistic equation was among the first to be discovered that show this type of behavior.

In conclusion, three different *modeling paradigms* produce three very different types of behaviors, even though the phenomenon being described is the same. This is a situation that mathematical modelers often encounter, and good modeling requires an understanding of both the phenomenon being modeled, and of the tools used.

The top ten: Continued from front cover

1.2 Devise a plan

This is usually the hardest part, because it often depends on an unexpected revelation or on recognition of a similar problem you've encountered in the past. Indeed, most of the rest of the article will be devoted to tricks that help you come up with a plan. When doing this, it is important to recognise the difference between a mathematically simple approach that is computationally complicated (often called "brute force") and an inspired, efficient, "elegant" solution. Often a brute force solution will occur to you very quickly, while you may have to think long and hard to come up with an elegant one.

When solving questions for practice or pleasure, always seek the elegant answer because, aside from being more satisfying, it is also more likely to reveal an idea that can be used later. In a time limited contest environment, it is necessary to weigh the amount of work involved in a brute force approach against the probability of finding an elegant solution.

1.3 Execute the plan

Go to it! Write up the proof, or do the calculations. Along the way, look for as many ways as possible to check the accuracy of your work: do my answers work when substituted into the original equations? Do my solutions behave correctly in extreme or degenerate cases ($x \to \infty, \ldots$)?

1.4 Revise and reflect

In a high-pressure contest, omit this step, but it is the only way to accumulate a toolbox of helpful techniques to use in future. Once you've finished, ask yourself: am I happy with my solution? Was there a better way? What did I learn that might be helpful in another problem? The more problems you do, and the more you reflect on them, the better you will be at problem solving.

2 A few of my favourite things

Knowing the basic formal method of problem solving is important, but it isn't much help when you've run into a hard problem, the clock is ticking, and you don't know how to proceed. To make much progress, it is essential to know a few basic methods that can be adapted to many problems with a little cleverness. Below are the ten techniques that, in my experience, are the most useful on high school math contests. They are arranged approximately in order of difficulty: most of them are useful at all levels, but the last few tend only to appear on more difficult long answer papers.

2.1 Periodic sequences

In many cases, by writing down the first ten or so terms in a sequence, it is possible to observe a pattern. For example: (2003 Fermat [2]) A sequence of numbers has 6 as its first term, and every term after that is defined as follows: If a term t is even, the next term after that is $\frac{t}{2}$; if a term s is odd, the next term is $3s+1$. Find the 100^{th} term in the sequence. The first few terms in the sequence are:

$$\begin{array}{llll} a_1 = 6 & a_4 = 5 & a_7 = 4 & a_{10} = 4 \\ a_2 = 3 & a_5 = 16 & a_8 = 2 & a_{11} = 2 \\ a_3 = 10 & a_6 = 8 & a_9 = 1 & a_{12} = 1 \end{array}$$

The sequence repeats starting with the 6^{th} term, after which $a_{3n+1} = 4$, and $a_{3n+2} = 2$, and $a_{3n+3} = 1$, $n \geq 2$. Therefore, $a_{100} = a_{3(33)+1} = a_{3n+1} = 4$.

2.2 Telescoping Series

When asked to find the sum of a series, even if it is finite, it is best to have a more reliable method than typing each term into a calculator: one of the most useful is called telescoping. It involves finding some way of decomposing the terms of a series to get massive cancelling. Imagine that you are asked to find the sum of:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots + \frac{1}{10100}.$$

Note that this can be rewritten as

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \ldots + \frac{1}{100\times 101}.$$

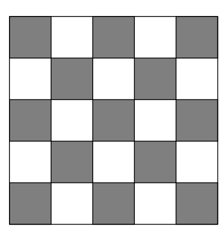
But $\frac{1}{1\times 2} = \frac{1}{1} - \frac{1}{2}$, $\frac{1}{2\times 3} = \frac{1}{2} - \frac{1}{3}$, and in general $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Thus the series can be rewritten this time as

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \ldots - \frac{1}{100} + \frac{1}{100} - \frac{1}{101}.$$

All the terms except the first and the last cancel, leaving $1 - \frac{1}{101} = \frac{100}{101}$.

2.3 Draw a diagram

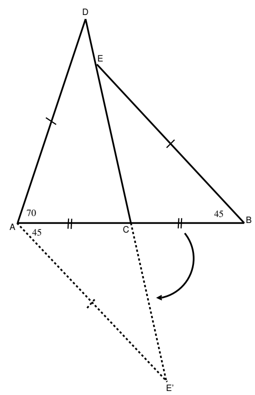
Whenever possible, drawing a diagram is a good idea, but sometimes particular insight comes from it, as in this problem from the Leningrad Problem Solving Circle [3]: 25 jealous neighbors live in a 5 floor apartment building with 5 suites on each floor. They all suspect that their neighbors living in adjacent squares (horizontally and vertically) live better than they do. Is it possible for all of them to move in such a way that they all end up in the apartments of one of their former neighbors?



Represent the apartment block as a 5×5 checker board. Every person is allowed to move up, down, left, or, right, which requires that every person starting on a black block must move to a white block and vice versa. In order for this to be possible, there must be an equal number of white and black squares, which is not the case (there are 13 black and 12 white). Thus there is no way of moving the people as required.

2.4 Imaginary scissors

Drawing a diagram is invariably recommended for geometry problems, but it is often beneficial to mentally rearrange components of the diagram to exploit symmetry or construct a new figure that is easier to analyse. This is the "imaginary scissors" technique, as my math teacher charmingly named it. Consider the part of the figure at right drawn in solid lines: $AC = CB$, $AD = EB$, $m\angle DAC = 70°$, and $m\angle EBC = 45°$. Find $m\angle ADC$ (1999 Manitoba Grade 12 Contest [4]). Without any further construction on the figure, this is a fairly time-consuming problem, but if you "cut" the figure along DC and paste ΔBCE onto the bottom of ΔADC as shown by the dotted line, you get an isosceles triangle! Then $m\angle DAE' = (70 + 45)° = 115°$, so $m\angle ADC = \frac{180° - 115°}{2} = 32.5°$.



2.5 Funny factoring

There are many standard types of factoring covered in high school math, but contest questions tend to demand more innovative approaches. On a contest, virtually any expression of degree ≥ 3 that needs to be factored will necessarily have some simplifying trick: often some easily guessed factor like (x-1) divides it, or it can be factored by grouping. In some cases, adding the same quantity to both sides of an equation, perhaps when completing the square, will render both sides easily factorable by conventional means, as in the following question from the 2003 Manitoba Grade 12 Contest [4]: Find the roots of the equation $x^4 - 7x^2 = 4x - 5$. Adding $x^2 + 9$ on both sides of the equation simultaneously completes the squares of the LHS and RHS, giving

$$\begin{aligned} x^4 - 6x^2 + 9 &= x^2 + 4x + 4, \\ (x^2 - 3)^2 &= (x + 2)^2, \\ x^2 - 3 &= \pm(x + 2). \end{aligned}$$

This gives two quadratic equations, $x^2 - 3 = x + 2$ and $x^2 - 3 = -(x + 2)$, which can be solved easily to give $x = \frac{1 \pm \sqrt{21}}{2}$ and $x = \frac{-1 \pm \sqrt{5}}{2}$.

2.6 The AM-GM-HM inequalities

Inequalities are helpful whenever it is necessary to find the maximum or minimum value of an expression—something usually thought of as the province of calculus. The AM-GM inequality is the most common, and many inequalities can be proven by its judicious application. It states that the arithmetic mean of n positive numbers is greater than or equal to their geometric mean, which in turn is greater than or equal to their harmonic mean, with equality occurring when all the numbers are equal:

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}.$$

For example, imagine you are asked to minimize the area of the rectangle defined by the x-axis, the y-axis, and any point on the line $y = \frac{1}{x^2+1}$, with $x \geq 0$. If the width of the rectangle is x, its height will be $\frac{1}{x^2+1}$, so the area function is $A(x) = \frac{x}{x^2+1}$. Rearranging, $A(x) = \frac{x}{1+x^2} = \frac{1}{\frac{1}{x}+x}$, and applying the AM-GM inequality, $\frac{x+\frac{1}{x}}{2} \geq \sqrt{x \times \frac{1}{x}} = 1$. If $\frac{x+\frac{1}{x}}{2} \geq 1$, then $\frac{1}{\frac{x+\frac{1}{x}}{2}} \leq 1 \Rightarrow \frac{1}{\frac{1}{x}+x} \leq \frac{1}{2}$. Therefore, $A(x) = \frac{x}{1+x^2} \leq \frac{1}{2}$.

2.7 Proof by contradiction

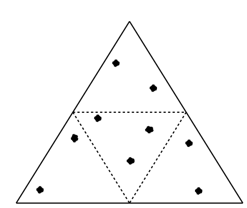
Begin by assuming that the statement you want to disprove is true. Then, by a series of logical deductions, show that this assumption leads to an irreconcilable contradiction: if every step along the way is logical, then the problem must be with the original assumption, which is thus shown to be false. Consider this problem from the 2001 Canadian Open [5], where a contradiction argument is used that deals with the parity of integers: Let $g(x) = x^3 + px^2 + qx + r$, where p, q, and r are integers. Prove that if $g(0)$ and $g(-1)$ are both odd, then the equation $g(x) = 0$ cannot have three integer roots. Assume that the equation $g(x) = 0$ has three integer roots, a, b, c; so it can be written as $g(x) = (x - a)(x - b)(x - c) = 0$. Expanding, obtain $g(x) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$. Substituting 0 and -1 for x:

$$\begin{aligned} g(0) &= -abc, & (1) \\ g(-1) &= -1 - (a + b + c) - (ab + ac + bc) - abc. & (2) \end{aligned}$$

From (1), and since $g(0)$ is odd, we know that abc is odd, so a, b, and c are all odd. Equation (2) is thus a sum of 8 odd numbers, which we know is an *even* number. But we are given that $g(-1)$ is an *odd* number, so we have a CONTRADICTION! This means our assumption was wrong, so $g(x) = 0$ cannot have three integral roots.

2.8 The Pigeonhole principle

The Pigeonhole Principle states that, if $nk + 1$ objects (like pigeons) are placed in n groups (pigeonholes), then one group must contain $k + 1$ objects. The following example [4] illustrates a standard manner in which it is used. Nine points, no three collinear, are located inside an equilateral triangle of side 4. Prove that some three of these points are vertices of a triangle whose area is not greater than $\sqrt{3}$. Begin by dividing the triangle up into 4 smaller triangles as shown, each of area $\sqrt{3}$. By the



Pigeonhole Principle, when placing nine points in four regions, one region must contain three points as shown in the diagram. Since the area of each region is $\sqrt{3}$, the area of the triangle defined by the three points in the same region will be less than or equal to $\sqrt{3}$, as required.

2.9 Modular arithmetic

This topic and the next require extensive explanation exceeding the space available for this article, but interested students are encouraged to investigate them on their own. In a nutshell, modular arithmetic deals with the remainders of integers on division by other integers.

2.10 Mathematical induction

Mathematical induction encompasses a whole new way of proving things. It works to prove that a statement is true for all numbers or objects in an ordered set in a fashion analogous to a line of dominos: to prove that all the dominos will fall, you must first establish that the first one can be pushed over, and then that the falling of a given domino will cause the domino after it to fall as well. With induction, the statement we wish to prove must be true for the first case (called the Base Step), and then if, by assuming it is true for the n^{th} case, we can prove it is true for the $(n+1)^{st}$ case (the Inductive Step), then the statement is true for all members of the set by induction.

The list presented above is by no means exhaustive, and I encourage you to add your own ideas to it. After all, the best approaches are those you devise yourself! For anyone interested in pursuing problem solving, there are a host of resources available; those listed as references are a good place to start. Whatever your level of experience, problem solving can be both an intellectually rewarding challenge and an amusing pastime. Have fun!

References

1. George Polya. *How to Solve It, 2^{nd} Edition.* Princton Unversity Press, 1971.

2. Past University of Waterloo contests are available online at http://cemc.uwaterloo.ca, or they can be obtained in print in the books *Problems, Problems, Problems Vols. 1-7*, Waterloo Mathematics Foundation, for sale at this website.

3. Ravi Vakil. *A Mathematical Mosaic.* Brendan Kelly Publishing Inc., 1996.

4. To my knowledge, a compilation of the Manitoba Grade 12 Contests has never been published. Ask your math teacher for old copies of them.

5. http://cms.math.ca/Competitions. This is a good source for more difficult Canadian Olympiad problems as well.

Problem Corner

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Dear Readers:

Welcome once again to the PROBLEM CORNER. Here is the problem from the last column and its solution: Members of the math club at your school are either teenagers or twelve year olds. When their ages are multiplied together, the product is 163,762,560. How many members does the club have? Would it be possible for the product to remain the same if the club contained only teenagers?

We begin with the prime factorization

$$163,762,560 = 2^7 \cdot 3^9 \cdot 5 \cdot 13.$$

Since ages of members must be between 12 and 19, there must be exactly one 13-year old and one 15-year old. This leaves seven 2's and eight 3's to be combined into members with acceptable ages. A single 3 must be combined with two 2's to give 12, and two 3's must be combined with a single 2 to give 18. A little experimentation shows that the only way to do this is to have two 12-year olds and three 18-year olds. The club therefore has seven members.

It is not possible to have the product remain the same if the club has only teenage members. The seven 2's and eight 3's would have to be combined into 18-year olds, requiring one 2 for each pair of 3's.

I am happy to report that this problem was successfully solved by David Newsom at Mennonite Brethren College. Congratulations David! I hope to receive more submissions from you and other students.

Send submissions on the next problem to:

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Here is your new problem: A straight stretch of railway track is one kilometre long. Its ends are fastened down so that they cannot move, but the remainder of the track is not fixed. Although this is not true, assume that the track is one continuous piece of steel with no breaks. Suppose now that the track is cut at its midpoint and a piece of track one metre long is added there. The ends of the track do not move, and the track bows into the arc of a circle. How far away from its original position is the middle point of the track? Before you solve the problem, make what you feel is a reasonable estimate. Is it approximately one millimetre, one metre, or perhaps twenty metres? You may be surprised. You will need a calculator to obtain a final answer.