MANITOBA MATH LINKS

UNIVERSITY OF MANITOBA

Volume II, Issue #2, Winter 2002

University of Manitoba Outreach Project

Published by the Department of Mathematics

Probability: When the Odds are Not Obvious

Dean Slonowsky

Each week, countless Canadians purchase 6/49 tickets in the hope that a $1 investment will net a multi-million dollar payoff. What are the chances of winning (or at worst, sharing) the jackpot by playing a single ticket? This depends on the number of different tickets, i.e., "outcomes" that are possible in such a draw. How many different outcomes are there? As many as the number of ways we can select 6 distinct numbers (in any order) from among the numbers 1 through 49. This is an example of a "combination", and this particular combination, denoted ${}_{49}C_6$ equals 13,983,816. Thus, the odds of winning are 1 in 13,983,816; next to impossible! For a comparison, the odds of rolling a 3 on a six-sided die is 1 in 6. ("Die" is singular for "dice".) How can you increase your chance of winning the 6/49 jackpot? Buy more tickets, of course! Indeed, if you purchased 13,983,816 tickets, each with a different combination of 6 numbers, you would be guaranteed to win! (How could you lose?) This would even ensure a profit, so long as the jackpot was over $13,983,816 and you were the sole winner.

Of course, buying one ticket for each and every combination (filling in the 6 dots on each of the 14 million or so tickets) is too time consuming. One way to save time would be to buy these 13,983,816 tickets using "Quick Pick". Quick Pick automatically selects a ticket for you. To accomplish this, the computer in the ticket dispenser selects one of the 13,983,816 combinations at random. Just like consecutive rolls of a die, the outcome of each Quick Pick selection does not depend on the outcome of any other Quick Pick selection. This randomness and "independence" are paramount to fairness in the Quick Pick scheme. What are the odds of winning the jackpot by playing 13,983,816 Quick Picks?

A little thought suggests that you are no longer guaranteed to win. Indeed, if two of your Quick Picks had the same 6 numbers (could happen), you would miss out on one possible combination. If that missing combination was the one drawn by the lottery, you would not win the jackpot! In fact, hundreds, perhaps thousands, of your Quick Picks might have matching combinations. How will this affect your odds of winning? Here, we can no longer rely on intuition alone. We must also employ some basic results from *probability theory*. Probability is a branch of math which enables us to model and analyze situations with uncertain outcomes, lotteries being a prime example.

Like any other area of math, probability has its own language by which to express various concepts. There are two key ingredients in Probability. *Events*, denoted by capital letters, *A*,*B*, etc. represent things which can possibly occur, for example,

$$A = \text{“roll a 3 on next toss of a die”} .$$

The second ingredient is a *probability measure P* which assigns, to any event *A*, a number *P(A)* between 0 and 1. A value of *P(A)* near 1 is interpreted as *A* being very likely to occur, indeed, certain to occur if *P(A)* = 1. On the other hand, If the value of *P(A)* is close to zero, this is interpreted as *A* being very unlikely (impossible, if $P(A) = 0$). $P(A) = \frac{1}{2}$ means *A* is as likely to occur as it is likely not to occur. For example, if we flipped a "fair" coin, then $P(\text{heads}) = P(\text{tails}) = \frac{1}{2}$.

How do we translate our "odds of" notion of likelihood to the probability notion? If the odds of *A* happening are 1 in *M*, this corresponds to $\mathrm{P(A)} = \frac{1}{\mathrm{M}}$.

Since it is reasonable to assume the outcomes of two consecutive coin flips are "statistically independent",

$$P(\text{two heads in a row}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} .$$

(continued on page 3)

MANITOBA MATH LINKS WEBSITE

.......www.umanitoba.ca/faculties/science/mathematics

A NOTE FROM THE EDITORS:

We welcome comments from our readers and value their advice. Do you have any suggestions for improving Math Links? Topics for new articles? Drop us a line....either by e-mail or regular mail....to the attention of our Co-ordinator. We enjoy hearing from you...!

SPEAKERS AVAILABLE

If you are interested in having a faculty member come to your school and speak to students, please contact our newsletter co-ordinator.

DATES TO MARK ON YOUR CALENDAR:

Information Days at the U OF M: February 12th & 13th

The Math Links Newsletter is published by the Mathematics Department Outreach Committee three times a year (Fall, Winter & Spring) .

EDITORS:	*Prof. T. G. Berry* *Prof. R. Craigen* *Prof. A. Gumel* *Prof. R. Padmanabhan*
CO-ORDINATOR:	*S. Kangas* *(kangass@cc.umanitoba.ca)*
OUR WEBSITE:	*www.umanitoba.ca/faculties/science/ mathematics*
MAIL ADDRESS:	*Math Links Newsletter* *Department of Mathematics* *University of Manitoba* *342 Machray Hall* *Winnipeg MB R3T 2N2*
FAX #:	*(204) 474-7611*
TEL #:	*(204) 474-8703*

COOL WEBSITES TO EXPLORE

R. Padmanabhan

Welcome to 2002. We are living in the age of Internet and the www is brimming with interactive math sites. In this issue we choose to mention some sites dealing with the topic of tiling the Euclidean plane with non-periodic tiles.

It is easy to cover (or tile) the plane using regular shapes like equilateral triangles, squares, or even regular hexagons. Here the basic pattern keeps repeating and hence these are examples of the so-called periodic tilings.

In 1974, Roger Penrose, a math professor at Oxford solved the following problem: What is the smallest number of different tiles needed to cover the plane *without creating* a repeating pattern? Believe it or not, one needs only two tiles to create a non-repeating tiling of the plane! Penrose used a kite and a dart to tile the plane. They looked somewhat like the diagram on the bottom of page 3.

Penrose tilings have made a big splash in the math world in recent years, but you may be asking yourself what the excitement is all about? Aside from being beautiful, Penrose tiles are interesting because they always tile the plane nonperiodically, even though they can be constructed from just two tiles, following a few simple rules. This caught everyone by surprise, because you would think that such a tiling would turn out to be very symmetric, like the usual wallpaper tilings. But they are not.

http://www.scienceu.com/geometry/articles/tiling/penrose.html
http://galaxy.cau.edu/tsmith/KW/goldenpenrose.html
http://www2.spsu.edu/math/tile/defs/definitions.htm
http://freeabel.geom.umn.edu/apps/quasitiler/about.html

Some of these are interactive sites. You may be able to design your own pattern. Have fun.

TABLE OF CONTENTS:

The notion of statistical independence and the corresponding product rule can be extended to any number of events; given statistically independent events $B_1, B_2, \cdots B_n$,

$$P\left(B_1 \text{ and } B_2 \text{ and...and } B_n \text{ occur}\right) = P(B_1)P(B_2)...P(B_n) \qquad (1)$$

Now, let's revisit our original problem using the basic results developed above. Define A to be the event, "win 6/49 jackpot with 13,983,816 Quick Picks" and B_i to be the event, "ticket i does not win the jackpot", defined for each of our tickets, labelled from $i = 1$ up to $i = 13{,}983{,}816$. Let A' be the event that we do not win the jackpot. Some thought leads to A' = "lose on every ticket" = "B_1 and B_2 and $\cdots$ and $B_{13,983,816}$ occur".

Therefore, since A' is the opposite of A,

$$\begin{aligned} P(A) &= 1 - P(A') \\ &= 1 - P(B_1)\,P(B_2)\cdots P(B_{13,983,816}) \qquad \text{[by (1)]} \\ &= 1 - \left[1 - \frac{1}{13,983,816}\right]^{13,983,816}. \end{aligned}$$

This does not give a clear answer, being essentially of the form "1 minus 1^{∞}". However, using limits it can be shown that

$$\left(1 - \frac{1}{n}\right)^n \approx \frac{1}{e}$$

for large n, where $\approx$ means "equal up to many decimal places" and e is *Euler's constant*, the famous irrational number, 2.711828... Therefore, since $n = 13{,}983{,}816$ is large, we can write

$$P(A) \approx 1 - \frac{1}{e} = 0.6321...$$

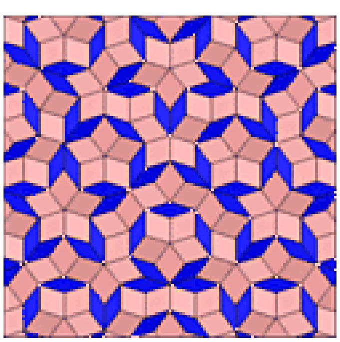
(accurate well beyond 4 decimal places). Paradoxically, $P(A)$ is quite far from 1, despite our having so many tickets! You would have a better chance rolling one die and not getting a 1 or a 6; the probability of that event is:

$$\frac{4}{6} = 0.66...$$

Certainly not worth risking nearly $14 million dollars!

This is just one example illustrating how the formal rules of probability enable us to calculate in a very precise way the likelihood of a complex event. Probabilists (the author included) work on developing advanced techniques for the evaluation of probabilities and apply these techniques to complex events encountered in areas such as physics, genetics, finance and astronomy, just to mention a few.

****



Example of Penrose Tiling

Mathletics at U of M

R. Craigen

Have you ever thought of Math as a competitive sport? It may come as a surprise to know that we have a Math team at the University of Manitoba. These tough competitors represent our school well.

I have coached our math team for the last three years, and have been joined by Dr. K. Kopotun for the last two. It is a real privilege (and a real challenge!) to work with such sharp students and to try to keep them on edge. Each year we participate in two major undergraduate mathematics competitions:

The ***Team Competition*** sponsored by the North Central Section of the Mathematical Association of America involves universities in our geographical region, which includes Manitoba, Saskatchewan, North and South Dakota, Minnesota, and surrounding region.

In this contest, which takes place in the middle of November, teams of up to three students work together on a set of ten problems for three hours, submitting a single set of solutions. This year 64 teams from 22 schools participated.

U of M fielded four teams for this event, all of which placed in the top third of the rankings, as follows:

Team TBA : Matt Hasselfield, Tim Nikkel and Xi Wang, placed (tied for) second overall.
Metrically Spaced : Nick Harland, Dan Potter and Roger Woodford, placed fourth.
The Gears : Fan Chan, Vincent Cheung and Dan Shen, placed 20th.
The Irrational Epsilon Deltas: Karen Johannson, Manon Mireault, and Mercedes Scott, placed 21st.

Will Gibson, another U of M student, also participated individually—but as an unofficial team; had he been ranked according to his score he would have single-handedly placed 27th.

We have had two second-place and two fourth-place finishes in the three years we have been competing in this event (next year, perhaps, we'll win it all!).

The other two Manitoba Universities that participated also had very respectable showings; a University of Winnipeg team placed 12th and one from Brandon placed 17th.

The ***Putnam Examination*** is a North-America-wide individual event that involves upward of 3000 elite undergraduate math students each year from some 500 schools competing for money, scholarships and glory. Many consider the Putnam Exam to be the ultimate undergraduate mathematics contest—the BIG one.

The exam consists of two 3 hour sessions on the same day (the first Saturday in December). In each session students are given six questions to solve. For some idea of the difficulty of this exam it helps to know that, out of 120 possible points (10 for each question), in some years over half the students get 0.

(continued on page 4)

Each university selects three students whose scores are compiled after the contest to form a team score for the school. This year eleven U of M students wrote the Putnam Exam:

Robert Borgerson,
Matt Hasselfield,
Nicholas Harland,
Karen Johannson,
Craig Kasper,
Tim Nikkel,
Dan Potter,
Danny Shen,
Xi Wang,
John Wedgewood,
Roger Woodford.

The results will not be known until next March. In the last couple of years we have had a couple of students place in the top 500, one in the top 200, and attained team scores in the top 100. We hope for a top 50 placement in years to come, which would place us solidly among the elite math schools in North America.

How do we train for competition? This year we met twice weekly to pose and solve problems together. Our discussions can get very lively as we examine solutions from all sides, trying to find improvements and looking for logical gaps. We have all-day practice competitions the Saturday before each of the two contests, writing sample tests in the morning and competing in a game-show format in the afternoon, for what we have (ironically) referred to as "fabulous prizes". This year a team from the University of Winnipeg joined us for the event, and in years to come we plan to make these practices an all-Manitoba event.

Math at this level is also a social event. On competition days, students are treated to lunch at local restaurants, and we dissect the day's competition together. For the all-day training sessions we order lunch in so that our activities can continue.

While undergraduate contests may involve university mathematics, they often contain problems that anyone, suitably motivated, might attempt, regardless of their background. Here, for example, is one such problem from this year's NCS/MAA team contest:

The *fractional part* of a number in decimal representation is the part that comes after the decimal. For example, the fractional part of 3.125 is 0.125, and the fractional part of π is 0.14159... You are to find a positive number r, such that the sum of the fractional parts of r and $1/r$ is 1.

Here is one from this year's Putnam Exam:

The numbers 1 to n^2 are arranged into a square, from 1 to n along the first row, $n+1$ to $2n$ along the second and so on. Each number is assigned a color, red or black, in such a way that half the numbers in each row, and in each column, are red and the other half black. You are to prove that the sum of the red numbers must be equal to the sum of the black numbers.

WARNING: These problems can be attacked by elementary means, but this does not mean that they are easy to solve. They can also be addictive!

Do you enjoy high school math competitions? Each spring our department cooperates with the local Institute of Industrial Mathematical Sciences in hosting a four-week problem-solving workshop for keen, competitive high school math students. Many of our math team members help out at this workshop, so it is a chance for you to glimpse what is ahead, and to prepare for contests at your own level.

If University Mathletics is attractive to you, plan on getting involved on our team in your first year here, and participate in as many different high school math competitions as you can. The more preparation and experience you have, the greater the chances of your success. Unlike some other varsity sports, we do not restrict participation — students of all strengths and programs of study are welcome to (and do) join us. Our best score in last year's Putnam exam was by a computer science student; one of our NCS/MAA teams this year consisted of engineering students; we have some other very serious competitors who are not strictly math students. Spirit, creativity, competitiveness and willingness to learn are the most important characteristics for this activity.

For more information on our competitive mathematics activities visit our departmental web pages, *www.umanitoba.ca/faculties/science/mathematics*, or email me at *craigenr@cc.umanitoba.ca.*

****

An astronomer is on an expedition to darkest Africa to observe a total eclipse of the sun, which will only be observable there, when he's captured by cannibals. The eclipse is due the next day around noon. To gain his freedom he plans to pose as a god and threaten to extinguish the sun if he's not released, but the timing has to be just right. So, in the few words of the cannibals' primitive tongue that he knows, he asks his guard what time they plan to kill him.

The guard's answer is "Tradition has it that captives are to be killed when the sun reaches the highest point in the sky on the day after their capture so that they may be cooked and ready to be served for the evening meal."

"Great," the astronomer replies.

The guard continues, though, "But, because everyone's so excited about it, in your case we're going to wait until after the eclipse."
Edward Ruden

****

An Interesting Combinatorial Problem and CONTEST

Professor Y. Nought

In this article, we present a rather interesting "counting" problem (more properly known as a combinatorial problem), together with a geometrically-motivated algebraic derivation of the answer to the question posed later in the problem.

For the purposes of the following discussion, let n and N denote positive integers, fixed but arbitrary, as defined below:

Let n denote the number of mutually non-parallel straight lines to be drawn in a plane. Let N denote the corresponding maximal number of regions into which the plane is subdivided by the above n lines.

Two important observations concerning the above definitions for n and N should be noted :

The fact that the lines are to be mutually non-parallel guarantees that each pair of lines shares precisely one point of intersection. The requirement that the number of regions created be maximal avoids the "degeneracy" that occurs when more than two lines share a single common point of intersection.

The problem to be considered may be simply expressed as ***"For each positive integer value of n, find the corresponding positive integer value of N"***, or even more succinctly as ***"Find N as a function of n ".*** (Although the problem has now been expressed in an algebraic form, one should not forget the geometry associated with the definitions of n and N, as it will help us solve this problem.)

Although mathematics is typically not viewed as an experimental science, we now encourage you to "experiment" (by drawing representative diagrams) in order to confirm the validity of the following table of values:

n	N
1	2
2	4
3	7
4	11

Remark: Even at this early point in your mathematical career, you realize that it is only possible to draw line segments to represent lines and thus your diagrams are simply idealized representations of the "complete picture".

To determine N as a function of n, we observe that when there are precisely $n - 1$ lines which have previously been drawn (with the corresponding number of regions being $N(n - 1)$), the inclusion of the n^{th} line (which must now intersect each of the preceding lines precisely once, with multiple points of intersection disallowed in order to avoid "degeneracy") gives rise to an increase $\Delta N = N(n) - N(n-1)$ in the number of regions which is precisely $(n - 1) + 1 = n$.

Thus we may write the general recurrence relation

$$N(n) = N(n-1) + \Delta N = N(n-1) + n \quad \text{for} \quad n = 2,3...$$

or equivalently,

$$\begin{aligned} N(n) &= n + \big((n-1) + N(n-2)\big) \\ &= n + (n-1) + \big((n-2) + N(n-3)\big) \\ &\ldots \\ &= n + (n-1) + (n-2) + \ldots + 2 + N(1) \ . \end{aligned}$$

But $N(1) = 2$, so that $N(n) = 1 + \sum_{k=1}^{n} k$ in which the familiar expression ***for the sum of the first n positive integers*** , namely $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n$, appears. It is well known (you should probably look for a reference if you are unfamiliar with this fact) that this ***finite arithmetic series*** has sum $\frac{n(n+1)}{2}$, from which one may conclude that

$$N(n) = 1 + \frac{n(n+1)}{2} = \frac{1}{2}\left(n^2 + n + 2\right). \ .$$

For completeness, one should verify that this formula agrees with the data in the above table, and could also go on to compute $N(37)$, $N(432)$, etc.

A Contest:

The Challenge:

The MathLinks Editorial Committee hereby challenges **individual students, or teams of students,** to solve and present their solution(s) to a second combinatorial problem which, like the above problem, may be expressed in a geometrical form, but whose solution is combinatorial in nature and involves considerable algebraic manipulation.

The Problem:

Let C be a fixed circle drawn in the plane. Let D be the disk enclosed by this circle. Once again, let n and N denote (fixed but arbitrary) positive integers as defined below:

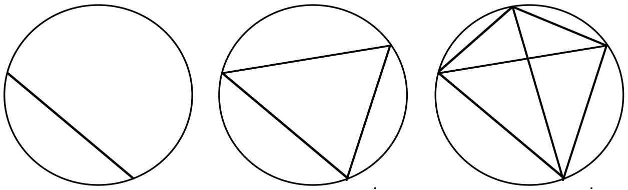
Let n denote the number of distinct points to be chosen on the boundary curve C of the disk D, and suppose that each of these points is connected to each of the remaining chosen points by a straight line segment. We shall henceforth assume that $n \geq 2$. Let N denote the maximal number of regions into which the disk D is subdivided by the above set of lines joining the chosen n points.

Once again it is noted that since the line segments are required to be straight, each pair of them must intersect in precisely one point (which in this case may be a common endpoint of the pair of line segments), and again the condition that the number of regions formed is maximal is included to avoid the degeneracy described previously.

(continued on page 6)

The problem may once again be expressed as ***"Find N as a function of n ".***

Below are shown representative diagrams in the cases when the number of chosen points n is 2, 3 or 4.



The prize:

The "winning" entry will be awarded one copy of each of the two textbooks:

Calculus A Complete Course (Robert A. Adams)
Elementary Linear Algebra (Venit/Bishop)

(These two books are currently the prescribed textbooks for the introductory calculus and linear algebra courses (136.150/170 and 136.130) offered by the Department of Mathematics.)

The rules:

1. Both individual and team entries will be accepted, and will be judged in a single competition. All entrants must be secondary school students, and may only contribute to a single entry.
2. All entries will be judged for completeness, accuracy and presentation.
3. Each entry must be submitted on behalf of the entrants by a teacher who is asked to attest to the fact that the solution and entry were prepared solely by the students in whose name the entry is made.
4. Entries should be sent to the Math Links Editorial Committee, c/o Department of Mathematics, University of Manitoba, Winnipeg, Mb R3T 2N2 and should be received by **4:00 p.m. on Friday , March 1st, 2002.**

The decision:

The decision of the judging committee will be final.
The "winner(s)" will be announced in the next edition of MathLinks.

SOME HINTS:

As in the above solved problem, it is suggested that a formula be developed for the increase $\Delta N = N(n) - N(n-1)$ in the number of regions which occurs when the n^{th} chosen point (together with the accompanying line segements) is added to the diagram in which the first (n - 1) chosen points have already appeared. This will result in a recurrence relation similar to (but more complicated than) that shown above. In addition, it may be helpful to seek out formulae for the ***sums of the positive integer powers of the first n integers,*** represented by the series

$$\left(\sum_{k=1}^{n} n^{\ell} \text{ with } \ell \text{ positive integer}\right).$$

****

When Bad Sums Happen To Good Fractions: Getting To The Root

R. Craigen

Many young scholars come to grief when they forget the rule for adding fractions. Let us review the correct procedure. To add 1/2 to 2/3, make equivalent fractions with a common denominator, 3/6 + 4/6, then add ONLY the numerators: (3 + 4) / 6 = 7/6.

Though it seems unnatural at first, the reason for finding comon denominators is obvious, upon reflection: a (rational) fraction *m/n* is simply an integer *(m)* number of “things” ((1/*n*)’s).

For example, 2/3 means two of the things called “thirds”; 1/2 is one of the things called “halfs”. Two things plus one thing is usually three things, but let us be wary of adding apples to oranges -- we have different kinds of “things” here. (By the way, in spite of the popular saying to the contrary, one CAN add apples and oranges; for example, (2 apples) + (3 oranges) = (5 fruit). But this piece of wisdom does not help us here...)

Our escape is to reinterpret each fraction so that they are both numbers of the same kinds of things. Conveniently, this is always possible -- by “finding a common denominator”. A useful procedure is born, and the rest is history.

Now let us consider the rule used by a poor scholar who somehow has gotten confused:

$$a/b + c/d = (a + c) / (b + d) \quad (!?)$$

(add the numerators and denominators separately). Of course we know this is wrong, so don’t ever do this! But -- HOW wrong can it be? Use the symbol • for our “bad sum”. For simplicity, and to avoid dividing by 0, we insist that *a, b,c* and *d* be positive integers.

Now use a bad sum on our initial example: 1/2 • 2/3 = 3/5; this is nowhere near the correct answer, 7/6. In fact, it’s about half!

Let’s try another, this time adding numbers of much different sizes: 1/4 • 17/3 = 18/7 ≈ 3, while the correct answer, 71/12 ≈ 6 -- still about half!

Wait a minute -- in both cases, we get about half of the (true) sum of the two fractions -- their average, or mean. Is this always the case? Unfortunately not. Consider *N*/1 and 1/*N*, where *N* is some large number; half the sum is about *N*/2, while :

$$N/1 \bullet 1/N = (N+1) / (1+N) = 1$$

-- not even close!

Still, there is something suggestive about our observation above...

In all our cases so far, we obtain a number between the two summands. Is the bad sum of two positive fractions always numerically between them?

Let us consider an arbitrary bad sum: $a/b \bullet c/d = (a+c)/(b+d)$. Suppose $a/b \le c/d$. Cross multiply: $ad \le bc$. Now add cd to both sides: *ad* + *cd* ≤ *bc* + *cd*. Dividing by *d(b + d)*, we obtain *(a + c) / (b + d)* ≤ *c/d*. Similarly we can show that the bad sum is greater than or equal to *a/b*.

To summarize: *the bad sum of two positive fractions lies numerically between them.*

Your reaction might be, "What a useless piece of information -- it still doesn't help us find the correct sum!" The latter statement may be true, but the former is not. Often doubtful looking ground leads to worthwhile insights, and this exercise is no exception!

Let our poor scholar now consider the problem of finding the square root of an integer, say a non-square, like 10. We know that, if an integer is not a perfect square, then its square root is not rational. Suppose, however, that we would be happy with a rational approximate square root. How shall we find one? It is easy enough to get an integer value; 10 = 9 + 1, so clearly the square root of 10 is a bit more than 3. But, suppose this is not good enough. What shall we do?

Suppose *k* is some estimate (good, bad, or indifferent) of the square root of *n*, and we wish to find a "better" estimate. If it was exactly right, then *k* = *n/k*; if *k* is close, then so is *n/k*. The product of the two estimates is *n* so, if one is too large, then the other must be too small. The actual root is somewhere between them. So, a number between them is an even "better" guess.

If *k* is rational, so is *n/k* -- and their bad sum is a rational number between them!

Let us estimate the square root of 10 with this idea: if 3 = 3/1 is close, then so is 10/3. A "better" pair of guesses, then, are *k* = 3/1 • 10/3 = 13/4, and 10/*k* = 40/13. Our first two estimates place $\sqrt{10}$ in the interval (3,3.33), and the second places it in the interval (3.07, 3.25). Better, but still not spectacular.

Repeat the process on the new interval: the new *k* is 13/4 • 40/13 = 53/17, while 10/*k* = 170/53, giving the interval (3.118, 3.208). Since $\sqrt{10} \approx 3.162278$, we are on the right track -- but proceeding slowly. Can we improve this method?

What happens to the bad sum if we use equivalent fractions, say *(sa) / (sb)* • *(tc) / (td) = (sa + tc) / (sb + td)*? The answer changes! Now we see that the bad sum is REALLY bad -- from the SAME two rational numbers we get DIFFERENT answers, by representing them as equivalent, but different, fractions .... but the bad sum is ALWAYS between the two "bad summands".

Maybe some pair of equivalent fractions gives a better estimate. Start over, including all possibilities; could some bad sum of *x* = *a/b* and *y* = *Nb/a* give the root of *N*, which lies between them. That is, can we find *s* and *t* so that (*sa*) / (*sb*) • (*tNb*)/(*ta*) = (*sa* + *Ntb*) / (*sb* + *ta*) = $\sqrt{N}$? A little algebra (spare us the details!) gives *s/t* = $\sqrt{N}$. Unfortunately, since $\sqrt{N}$ is irrational, there is no such *s* and *t*. Can we make *s/t* close? This is the ORIGINAL problem!

All we have so far is that a/b $\approx \sqrt{N}$.

Well, the best we can do, then, is take *s* = *a* and *t* = *b*, and obtain with a bit of algebra $\left(a^2 + Nb^2\right)/2ab$.

Try this with our working example. $\sqrt{10}$ is between (*a/b* =) 3/1 and (10*b/a* =) 10/3, so we consider: $(3\times3+10\times1)/(2\times3) = 19/6 \approx 3.166$, while $\sqrt{10} \approx 3.162$, a very good estimate! Now taking *a/b* = 19/6 (and 10*b/a* = 60/19), our next guess is:

$$\left(19^2 + 10\times6^2\right)/(2\times19\times6) = 721/228 \approx 3.162281$$

-- off by only about .000003!

In fact, our wild excursion has led us to a root finding method discovered (in a different fashion) by none other than Sir Isaac Newton; our poor scholar is in good company, indeed! Here are ideas to explore:

Try the method on a few of your favourite square roots.

- What about the problem of a good first guess? See what happens when you start off with a REALLY bad guess, like $\sqrt{13} \approx 7$. How many steps (if at all!) does it take before a good estimate is reached?

- We never showed that the bad fraction is greater than the smaller of the two numbers; do this (HINT: try adding something different in the middle step.)

- Explore the meaning of "between" in our assertion about bad sums. Could a bad sum be equal to one of the two summands, but not to the other?

- In what sense is it wrong to speak of the bad sum of two <u>rational numbers</u>, but okay to speak of the bad sum of two <u>fractions</u>? (HINT: this is a question about semantics; it has something to do with what happens when one of the summands in a bad sum is replaced with an equivalent fraction.)

- Another way to view the formula $\left(a^2 + Nb^2\right)/2ab$ is that it is the arithmetic mean of the two estimtes, *a/b* and *Nb/a*. Their <u>geometric mean</u> (the square root of their product) happens to be the exact number we are looking for. A famous inequality says that the arithmetic mean is never less than the geometric mean. What does this tell us about our estimate each time? Can you use this information to come up with an improvement on our method that is even closer -- without becoming unwieldy, of course?

****

As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality.
Albert Einstein (1879-1955)

****

Did You Know?????

Mathematics in Ancient Greece

Mathematics was comprised mainly of two subjects : Arithmetic and Geometry.

Arithmetic was taught at two levels . The first level was taught to middle and artisan classes and was based on calculation, an important skill needed by trades people. The second area, the science of numbers, was for the upper classes, who had the time and money to obtain a higher education.

Upper class students began their studies at home. Subjects included letters, music, gymnastics and a small amount of arithmetic or geometry. When boys were twelve years old, they were sent to a school where they were taught Grammar and the basics of Logic and Rhetoric. For most, this was the end of their formal education. Continued study involved either hiring a sophist[1], or attending one of the colleges or academies set up by people like Plato, Aristotle or Pythagoras.

Pythagoras's school began in Croton in 518 BC . Youths, of both genders, of the high aristocracy were admitted and studied a number of subjects including the science of numbers . This subject (the consideration of such things as perfect, abundant and square numbers and their properties) was the basis for the belief that everything can be mathematically expressed. It was here that music came to be considered as one of the Mathematical Sciences. Pythagorians practised a strict code of living based on the belief that the human soul could rise toward the divine through philosophical thought as a manner of purification. Many educationalists advised against prolonged consideration of mathematical ideas, fearing that it drew the mind away from the realities of the world causing a high level of abstraction. The political aims of the school raised up much opposition, and in a popular uprising in 497, the school was burnt.

Plato's Academy was established in 387 BC to educate future politicians and statesmen of Athens and is often described as the first European university. Plato's ideas of mathematics were far less extreme than those of Pythagoras . Mathematics was considered the basis from which to move into philosophical thought. Plato believed that mathematics provided the best training for the mind allowing students to understand relations that cannot be demonstrated physically. Logical thinking was prized in both philosophical discussions and in the political arena. Indeed, above the doorway to the academy were the words "Let no one ignorant of mathematics enter here".

Aristotle established the Lyceum in 335 BC which provided a much broader curriculum to Plato's Academy and dealt more with the natural sciences. Aristotle is credited with being the founder of the systematic study of logic. This school became known as the Peripatetic ("walking" or "strolling") school because much of the discussion took place while teachers and students were walking about the Lyceum grounds.

[1] *A member of a school of ancient Greek professional philosophers who were expert in and taught the skills of rhetoric, argument and debate, but who were criticized for specious reasoning.*

****

PROBLEM CORNER

D. Trim

Dear Readers:

Welcome back to PROBLEM CORNER. Here is the problem from the last column and its solution. A list of integers has mode 32 and mean 22. The smallest number in the list is 10. The median *m* of the list is a member of the list. If the list member *m* is replaced by *m* +10, the mean and median of the new list are 24 and *m* +10, respectively. If *m* is instead replaced by *m* - 8, the median of the new list is *m* - 4. What is *m*?

If the mean increases by 2 when *m* is replaced by *m* +10, and no other member is changed, there must be five members in the original list. Let them be $10 \le a \le m \le b \le c.$. The sum of the members of the list must be 5 times the mean 22; that is, $10 + a + m + b + c = 5(22) = 110$. The mode of the list is the most frequently occuring number. To say that the mode is 32 indicates that it occurs at least twice in the list. Neither *a* nor *m* can be 32 because the equation $a + m + b + c = 100$ would be violated. Thus, we must have $b = c = 32$, and this implies that $a + m = 36$. When *m* is replaced by *m* - 8, the new list is 10, *a*, *m* -8, 32, 32. Since the median is now *m* - 4, (and $m - 4 \neq m - 8$), it follows that *m* - 8 cannot be the median; *a* must be; that is, the list in ascending size is now 10, *m* - 8, a, 32, 32. Consequently, $a = m - 4$, and when this is solved with $a + m = 36$, the result is $m = 20$.

Here is your problem for next time: The floor function, or sometimes called the greatest integer function, is denoted by $\lfloor x \rfloor$. It is defined to be the largest integer not greater than *x* itself. For example, $\lfloor 4.2 \rfloor = 4$, $\lfloor 3 \rfloor = 3$, and $\lfloor -2.3 \rfloor = -3$. Show that the equation $x^{\lfloor x \rfloor} = 9/2$ cannot have a positive rational solution.

Recall that rational numbers are those that can be expressed as quotients of integers.

Let me encourage you to send a solution to:

S. Kangas,
Department of Mathematics,
The University of Manitoba,
Winnipeg, MB R3T 2N2

I will look at all submissions and print the names and schools of persons who solve the problem correctly and present it in a reasonable way.

****

Copernicus' parents: Copernicus, young man, when are you going to come to terms with the fact that the world does not revolve around you?!

Erin Leonard

****