MANITOBA MATH LINKS

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What Shape Is This Planet?

S. Kalajdzievski

The surface of the planet Earth is (roughly) spherical. This can now be easily verified; just use about $30,000,000 of your savings, buy a ticket to outer space, and take a peek. Things were much more complicated long time ago. The ancient philosophers who tried to infer such a global phenomenon as the shape of the earth could mostly rely on their local perception. Since the surface of the earth locally feels like the surface of a plane, the following problem was natural: what could the shape of the earth be, given that locally it feels like a plane at each and every point?

Thales (625-547 BC) figured out that the earth is indeed plane-like: he thought it was a disk floating on a flat body of water. That model had some obvious weak sides, spotted already by his contemporaries. After all, if the earth and the ocean it floats on are flat, why is it that the mast is the last part visible on a ship sailing to outer seas? Anaxamides (6th century BC) proposed a (possibly unbounded) cylinder with the land on the curved surface. Around the same time Pythagoras conjectured the spherical model of the surface of this planet.

So then, you are on the surface of an unknown planet; you look around and you see that it is more-or-less flat. What is its shape? Have we exhausted all possibilities with the above examples? If no, could we somehow classify all surfaces that locally feel like the surface of the plane (disregarding all laws of gravitation)?

We restrict the problem a bit, and establish our assumptions more precisely. Firstly, we assume that the surfaces (of the planets) we consider are bounded. A surface is bounded if it can be put inside a (possibly huge) box. This eliminates the plane itself; it also eliminates the unbounded cylindrical surface. Secondly, any two surfaces that can be deformed one into the other by stretching, contracting and bending (but without cutting or gluing) will be considered as being (essentially) the same. This allows us to disregard all wrinkles (mountains, valleys) on the surface; it also makes any ellipsoid (essentially) the same as the perfect sphere, as is illustrated in the following picture.

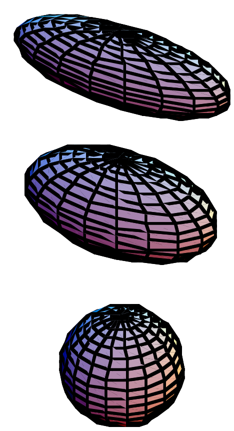

Figure 1. A simple deformation of an ellipsoid to a sphere (no cutting, no pasting).

Are there any bounded surfaces different from the sphere? Certainly yes!

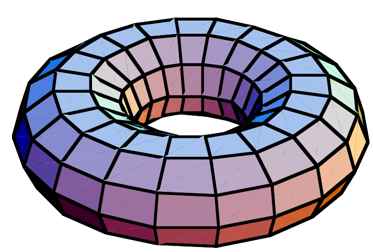

Figure 2. A doughnut

(continued on page 3)

MANITOBA MATH LINKS WEBSITE

.......www.umanitoba.ca/faculties/science/mathematics

A NOTE FROM THE EDITORS:

We welcome comments from our readers and value their advice. Do you have any suggestions for improving Math Links? Topics for new articles? Drop us a line....either by e-mail or regular mail....to the attention of our Co-ordinator. We enjoy hearing from you...!

SPEAKERS AVAILABLE

If you are interested in having a faculty member come to your school and speak to students, please contact our newsletter co-ordinator.

DATES TO MARK ON YOUR CALENDAR:

Problem Solving Workshop:
January 19 & 26
February 2 & 9

MANITOBA MATH LINKS

The Math Links Newsletter is published by the Mathematics Department Outreach Committee three times a year (Fall, Winter & Spring) .

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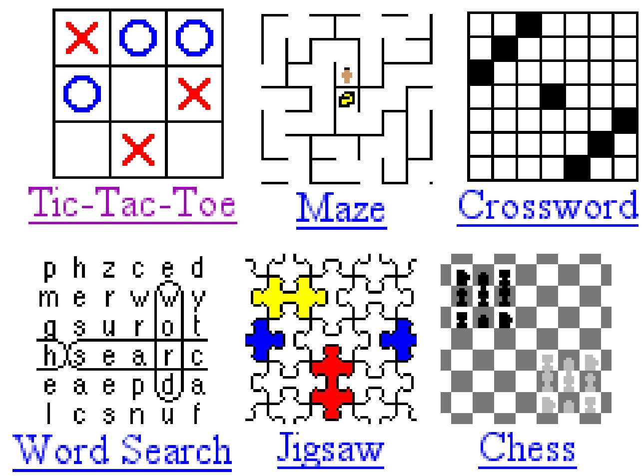
COOL WEBSITES TO EXPLORE

R. Padmanabhan

JIGSAW PUZZLE ON TORUS AND KLEIN BOTTLE



Now that you know what a torus is, you may like to play, jigsaw puzzles and other familiar games on the torus or similar exotic surfaces. For example, you assemble the puzzle pieces to form a complete picture not on a two-dimensional plane but on the surface of a torus etc. Sounds intriguing? Why not visit the following websites?

These sites describe several such surfaces obtained by gluing together sides of a paper strip in various fashions. Also, you can play online the above popular games right on your computer.
http://www.northnet.org/weeks/TorusGames/
http://www.cut-the-knot.com/do_you_know/paper_strip.html

TABLE OF CONTENTS:

(What Shape Is This Planet?...continued from page 1)

The surface of a doughnut (Figure 2) is called a torus. A torus *feels* locally like the surface of a plane, and an inhabitant on a torus-shaped planet (especially on the outer equator) would observe the same geometric phenomena as observed on the surface of the Earth. All the same, no deformation would transform a torus into a sphere (recall that no pasting or cutting is allowed). The last claim is intuitively plausible, but a precise proof needs precise definitions to start with (ours are not), followed by some not so easy mathematical work.

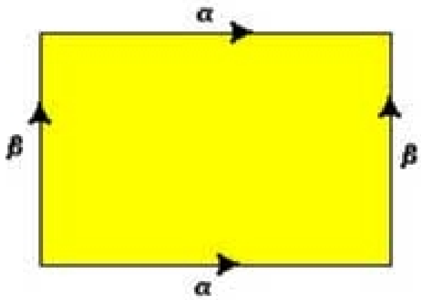


Figure 3. A torus from a rectangle

We can get a torus by gluing the edges of a rectangle as indicated in Figure 3. We glue the edges labeled by same letters in such way that the arrows in one pair of glued edges point in the same direction.

Exercises:

1. (easy) What surface is obtained by gluing all

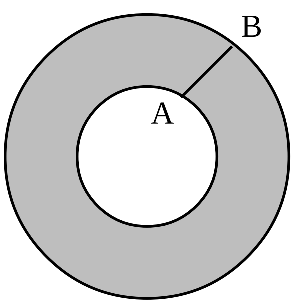


Figure 4.

radial points on the circular edges of the ring to the right (the points A and B in the illustration are radial)?

2. (not so easy) What do we get if we glue the

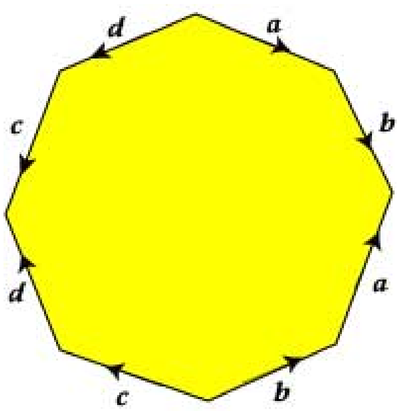


Figure 5.

indicated edges of the octagon Figure 5? You may want to dwell on this before proceeding further. At this point we need to distinguish two types of surfaces: orientable and non-orientable. A non-orientable surface is a surface possessing an orientation-changing path with same initial and terminal find itself at the same spot where it started from but with its leftpoint; a two dimensional ant taking a trip along that path would hand side turned to right hand and vice versa. An orientable surface is a surface that does not posses an orientation-changing path.

All of the previous surfaces are usual, orientable surfaces. For the time being we continue to focus on them: we are now ready to describe all different orientable bounded surfaces.

Theorem. Every orientable, bounded surface is either a sphere or a surface made of a finite number of tori (one or more) as Figure 6 (where we have used only four tori).

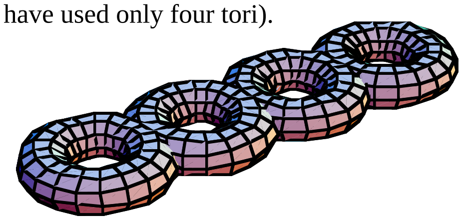


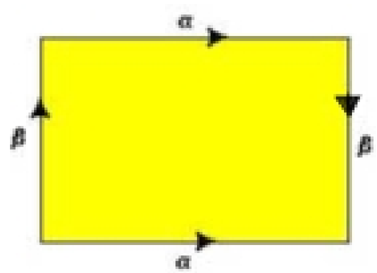
Figure 6. Connected sum of tori

The above tori (doughnuts) are connected by cutting out disk-shaped parts from two of them and gluing along the circles. The result is called a "connected sum of tori" (this sum has nothing to do with the usual summation of numbers). Connected sums of tori, together with the surface of the sphere, exhaust all possibilities for orientable surfaces of planets: if a planet feels like the plane and if we know it's surface is orientable and bounded, then it must be a sphere or a connected sum of tori.

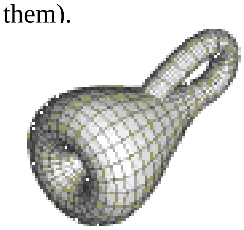
Naturally, you now wonder about non-orientable surfaces. Do such exotic surfaces exist?

They do! However, they are harder to depict, because they can not be fitted into three dimensions; these objects live in four dimensions. For the time being here is one that you might try to understand.

Take the elastic rectangular region as in Figure 3 above. Glue the vertical edges as earlier; the horizontal edges should now be glued in opposite direction (so the points at the right end of the upper edge should be glued with the points at the left end of the lower horizontal edge; see the picture below).



Can you visualize this procedure? The resulting surface is called a Klein bottle (see the picture below). Please note that Klein bottles have ***no*** self intersections (there is not enough space in 3 dimensions to avoid them).



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MATHEMATICS AND .....??? At The University of Manitoba

G. Woods

So, you say that you really like mathematics and you're pretty good at it, and you would like to take math at university..... BUT, there are other things that interest you too, and besides, you have to think about a career someday (at least they all keep telling you that). What can you do?

If you attend the University of Manitoba, there are lots of ways in which you can combine your enjoyment of math with your interest in other fields. Some of them you probably know about; others may come as news.

Let's start with a field that most people know about, namely Engineering. If you are technically inclined, and if you enjoy and are good at mathematics and physics, this may be the field for you. Engineering involves **lots** of mathematics. To oversimplify somewhat, electrical and computer engineering involve the most mathematics, civil engineering involves the least, and mechanical engineering is somewhere between.

A less widely known profession is actuarial science. Actuaries are the people who use mathematics and statistics to calculate what the premiums for insurance policies, pension plans, etc. should be. If you own a car, the size of your autopac premium was worked out by actuaries, using data about past experience with claims, etc.. Winnipeg has a big insurance industry (Great West Life has its headquarters here) and actuaries are in demand (and well paid).

There are several programs at the University of Manitoba that involve actuarial science. If you are oriented towards the business world, you might want to enter the Faculty of Management and study actuarial science there. If you like the more theoretical aspects of the subject, you can take an Honours program in Actuarial Science, or Actuarial Science and Mathematics, or Actuarial Science and Statistics, within the Faculty of Science.

Perhaps you like physics, but you are more interested in the scientific ideas than in the practical applications (like Engineering). To oversimplify again, physicists come in two flavours, namely experimental physicists (who work in labs and undertake experiments to learn more about the universe) and theoretical physicists (who use mathematics to construct theoretical "models" to explain the experimental results). Albert Einstein was a theoretical physicist. If theoretical physics appeals to you, or if you want to learn in detail about how mathematics can be used in a scientific field, you might want to enter the Faculty of Science and take a Joint Honours program in Mathematics and Physics. It contains approximately equal numbers of courses from each field.

Lots of people who like mathematics also like working with computers. In forty years or so computer science has grown from almost nothing to a huge field with many different aspects. Some of them involve very little mathematics, but some of the more theoretical aspects are very mathematical indeed. Mathematically oriented computer scientists (or, mathematicians with an interest in computing - these things blur at the edges) study things like "computational complexity" (i.e. how intrinsically complicated a computing problem is) and the design of computer algorithms. Maybe you have heard of Alan Turing, who was (in essence) a theoretical computer scientist in the days before computers! He was a leader of the team (which included many mathematicians) of British codebreakers in World War II that cracked the code that the German military command used to communicate with its U-boats and air force. Their efforts were crucial in winning the war. Who says that mathematics isn't relevant?

"Scientific computing" has become an important discipline in recent years. Mathematicians use computers to work out approximate solutions to really complicated equations that arise in engineering, physical and biological science, and finance - equations that are too difficult to solve exactly.

To prepare for a career in areas that blend computer science and mathematics in this way, you should take Joint Honours Mathematics and Computer Science within the Faculty of Science. People with this expertise are very much in demand.

Statistics is a math-related field devoted to drawing conclusions about past or future behaviour by a mathematical analysis of data. (If this sounds a lot like actuarial science, don't be surprised. Actuarial science is more specialized than statistics, but it makes use of a lot of statistics). Statistics has both theoretical and practical components; if you are interested in the theoretical aspects of the subject, you can take Joint Honours Mathematics and Statistics within the Faculty of Science.

Mathematical economics is that branch of economics that constructs mathematical representations, or "models", of the economy and uses them, in conjunction with economic data, to make predictions about how the economy will behave in the future, and what the effects of different policy options would be on the economy. You can be sure that Alan Greenspan (Head of the U.S. Federal Reserve Board, and in effect the chief economist of the U.S. government) studied lots of mathematics. Since the "laws" of economics are not as precise as those of physics or actuarial science, mathematical economics is not as sharp a tool as (say) theoretical physics. Nonetheless, it is used a lot, and its practitioners can have rewarding careers. We have Joint Honours Mathematics and Economics programs on the books, should you be interested in this field.

A related and growing field is the mathematics of finance. Big institutional investors in the stock market (pension plans, etc.) use computer programs to trigger decisions to buy or sell huge blocks of stock depending on the market's behaviour. Behind these programs are mathematical "models" of how the market will work; these have been prepared by mathematicians who specialize in the world of finance.

All of this just scratches the surface of professions that make heavy use of mathematics. But can't you study math all by itself, just for fun? Do you have to be interested in using it to do so something else? Not at all - many people will study mathematics (in a major program in the Faculties of Arts or Science, or in an Honours Mathematics program within the Faculty of Science) for its own sake, because it is fascinating to them.

(continued on page 5)

What happens to these people? A few become professors of mathematics at colleges and universities, and earn their living by teaching and doing research in mathematics. (Every year researchers discover new mathematical concepts and truths.) There soon will be lots of openings for qualified people who want to do this ("qualified" means possessing a Ph.D. in mathematics), because in a few years a majority of those who now teach mathematics at universities and colleges will be retiring.

More generally, specializing in mathematics at university can give you a way of looking at the world, and analyzing problems, that makes you valuable in many occupations. As an analogy, a highly conditioned athlete is in a position to become proficient at any one of a number of sports. A person who can write a clear, well organized persuasive essay will be an asset in a wide range of occupations that require good communication skills. Similarly, a person who understands university-level mathematics will be valuable in many technical and education related occupations.

For more information on math related careers and on choice of math courses in your first year at the University of Manitoba, go to our website at:

http://www.umanitoba.ca/faculties/science/mathematics/

then, under "Undergraduate Information", click on each of: Undergraduate Programs, Undergraduate Courses, Choose your Math courses, and Careers in Mathematics.

I hope we will see some of you in our programs in future years!

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A medallion issued by the Royal Mint in 1727 the year of Newton's death

ISAAC NEWTON (1642 - 1727)

Compliments of John Wiley & Sons, Inc.

As a youth he showed little evidence of his later brilliance, except for an unusual talent with mechanical devices - he apparently built a working clock and a toy flour mill powered by a mouse.

Because the plague was spreading rapidly through London, Newton, after graduating from Trinity College in Cambridge, returned to his home in Woolsthorpe and stayed there during the years of 1665 and 1666. In those two momentous years the entire framework of modern science was miraculously created in Newton's mind - he discovered calculus, recognized the underlying principles of planetary motion and gravity, and determined that "white" sunlight was composed of all colors, red to violet. For some reason he kept his discoveries to himself.

In 1667 he returned to Cambridge. In 1669 Newton succeeded his teacher, Isaac Barrow to the Lucasian chair of mathematics at Trinity, one of the most honored chairs of mathematics in the world. Thereafter, brilliant discoveries flowed from Newton steadily.

He formulated the law of gravitation and used it to explain the motion of the moon, the planets, and the tides; he formulated basic theories of light, thermodynamics, and hydrodynamics; and he devised and constructed the first modern reflecting telescope. Throughout his life Newton was hesistant to publish his major discoveries. In 1687, only after intense coaxing by the astronomer, Edmond Halley (Halley's comet), did Newton publish his masterpiece, "Philosophae Naturalis Principia Mathematica" (The Mathematical Principles of Natural Philosophy).

This work is generally considered to be the most important and influential scientific book ever written. In it Newton explained the workings of the solar system and formulated the basic laws of motion that to this day are fundamental in engineering and physics.

However, not even the pleas of his friends could convince Newton to publish his discovery of calculus. Only after Leibniz published his results did Newton relent and publish his own work on calulus.

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Annual High School Problem Solving Workshop

J.Brewster, I.I.M.S.
S. Olafson, I.I.M.S.

Each year, the Institute of Industrial Mathematical Sciences and the Department of Mathematics at the University of Manitoba jointly organizes a workshop for selected students in Senior 2, 3 and 4 from high schools in and around Winnipeg. The main purpose is to provide training for the Cayley, Fermat and Euclid Mathematics Competitions and to assist students in writing the Manitoba Senior 4 Competitions. In addition, the workshops also improve students' problem solving abilities, bring students with a passion for mathematics together to work individually and in groups, and allow the high school students to meet enthusiastic undergraduate students, graduate students and faculty members in the mathematical sciences.

The training is held in Machray Hall at the University of Manitoba, and consists of sessions on four consecutive Saturdays. The sessions run from 9:00 a.m. to 3:00 p.m., with lunch being provided to all participants during a one-hour break. These sessions are devoted to theoretical and practical aspects of problem solving. Problems are selected from various sources including past Cayley, Fermat and Euclid competitions. Class sizes are kept at reasonable levels and each class has more than one instructor to ensure fun is had by all participants.

Principals of high schools within a 50-km radius of the Fort Garry campus are asked to consult the Heads of their Mathematics Departments in the selection of one or more students for participation in the workshop based on each student's performance as an achiever, serious interest in the workshop, and commitment to participate in the workshop to the fullest extent. Interested students should also bring this to the attention of their mathematics teachers.

This year's workshop is to be held January 19 and 26, February 2 and 9, 2002.

For further information about the workshop, please contact the Institute of Industrial Mathematical Sciences at 474-6724 or iims@UManitoba.CA.

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100–180 = –80 and 64–144 = –80
So we have

$$100-180 = 64-144$$
$$100-180+81 = 64-144+81$$
$$10^2 - 2\times10\times9 + 9^2 = 8^2 - 2\times8\times9 + 9^2$$
$$(10-9)^2 = (8-9)^2$$
$$10-9 = 8-9$$
$$1 = -1 \text{ or } 2 = 0!$$

What went wrong?

If $x^2 = y^2$, then taking square roots we get either $x=y$ or $x=-y$. In our case, $10-9 = -(8-9)$ i.e. 1=1 and there is no paradox.

Visual "Proof" Of The Convergence Of Geometric Series

R. Padmanabhan

One of the most useful series is the geometric series $\sum_{k=0}^{\infty} r^k$. We find that for $r \neq 1$ we have

$$s_n = \sum_{k=0}^{n} r^k = 1 + r + r^2 + \ldots + r^n = \frac{1-r^{n+1}}{1-r} \textit{ for all } n \in N.$$

We also find that $\lim_{n\to\infty} r^{n+1} = 0$ when $|r| < 1$, so we conclude that $\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} s_n = \frac{1}{1-r}$ for $|r| < 1$. If $|r| > 1$, then $\lim_{n\to\infty} r^n$ does not exist (indeed r^n becomes unbounded in this case) and it follows that the series $\sum_{k=0}^{\infty} r^k$ diverges (has no sum).

There is a clever geometric argument that illustrates the convergence of the geometric series for $0 < r < 1$ as follows: Let $A = (0,0)$ and $B = (1,0)$ be points in the plane. Draw a line of slope r through A and a line of slope 1 through B. Since $0 < r < 1$, these lines intersect at a point, say C. Let P_1 be the point on the line segment $\overline{AC}$ that lies directly above B. Since $\overline{AC}$ has slope r and $\ell(\overline{AB}) = 1$, we have $\ell(\overline{P_1B}) = r$. ($\ell(\overline{AB})$ denotes the length of segment AB.) Let P_2 be the point on $\overline{BC}$ at the same height as P_1. Since the slope of $\overline{BC}$ is 1, $\ell(\overline{P_1P_2}) = r$. Continue in this manner with vertical and horizontal lines. If $\overline{CD}$ is the perpendicular from C to the line through A and B, then $\ell(\overline{AD})$ is the sum $s = 1 + r + r^2 + \ldots + r^3 + \ldots$. Since $\ell(\overline{BD}) = \ell(\overline{CD})$, we have $\ell(\overline{CD}) = s - 1$. Now the triangles ΔABP_1 and ΔADC are similar, so their corresponding sides are proportional. That is,

$$\frac{\ell(\overline{AD})}{\ell(\overline{AB})} = \frac{\ell(\overline{CD})}{\ell(\overline{P_{1B}})} \textit{ or } \frac{s}{1} = \frac{s-1}{r}$$

Solving for s we obtain the expected formula $s = 1 / (1 - r)$.

Page 7 contains a diagram that illustrates the above arrangement.

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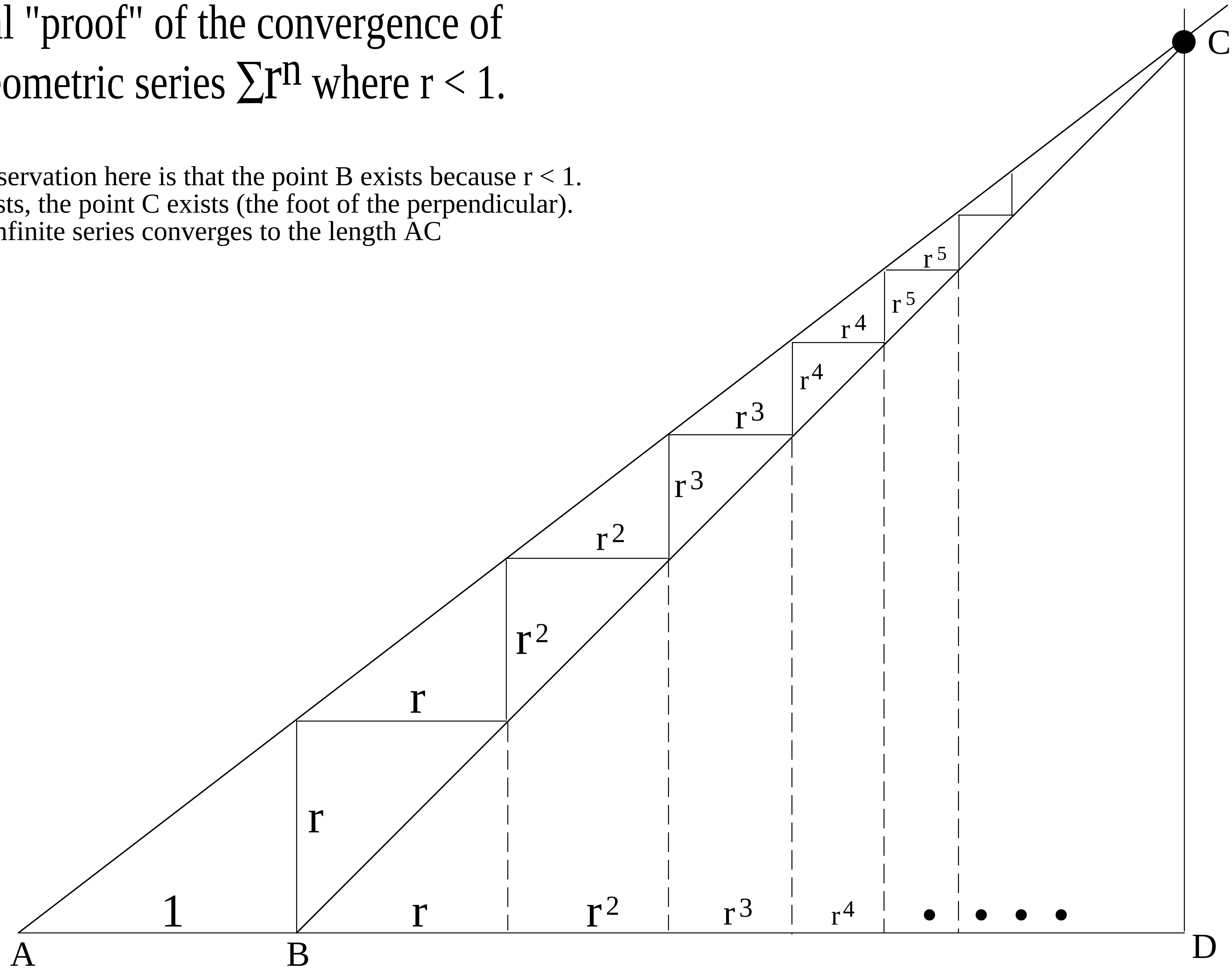
Visual "proof" of the convergence of the geometric series $\sum r^n$ where $r < 1$.

The crucial observation here is that the point B exists because $r < 1$.
Because B exists, the point C exists (the foot of the perpendicular).
The required infinite series converges to the length AC



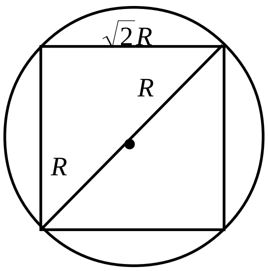
PROBLEM CORNER

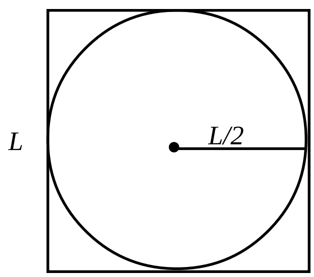
D. Trim

Dear Readers:

Welcome back to the PROBLEM CORNER. Here is the problem from Newsletter 3 and its solution. Inscribed in a circle is a square. A circle is then inscribed inside the square, a square in the circle, and so on and so on. What is the ratio of the sum of the areas of all the circles to the sum of the areas of all the squares?

The left figure below indicates that if we have a circle with radius R, the length of the side of the inscribed square is $\sqrt{2}R$. The right figure shows that when a circle is placed inside a square with side length L, the radius of the circle is $L/2$.





These two facts allow us to set up the radii for all inscribed circles and lengths of sides for all inscribed squares. Suppose we begin with a circle of radius R. The length of the side of the first inscribed square is then $\sqrt{2}R$. The radius of the circle inscribed in this square is $\sqrt{2}R/2 = R/\sqrt{2}$. The length of the side of the next square is $\sqrt{2}\left(R/\sqrt{2}\right) = R$. The radius of the next circle is $R/2$. The length of the next square is $\sqrt{2}(R/2) = R/\sqrt{2}$, and so on. It is perhaps best to place these in a table as shown below:

From the radii of the the circles, we can find the sum of the areas of all the circles,

$$\text{Sum of areas of circles} = \pi R^2 + \pi\left(\frac{R}{\sqrt{2}}\right)^2 + \pi\left(\frac{R}{2}\right)^2 + \pi\left(\frac{R}{2\sqrt{2}}\right)^2 + \ldots$$

$$= \pi R^2\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)$$

In parentheses, we have the sum of an infinite geometric progression with first term $a = 1$ and common ratio $r = 1/2$. Using the formula $a/(1 - r)$ for the sum of such a progression, we obtain for the sum of the areas of the circles,

$$\text{Sum of areas of circles} = = \pi R^2\left(\frac{1}{1-1/2}\right) = 2\pi R^2$$

Similarly, from the lengths of the sides of the squares, we can find the sum of the areas of all squares,

$$\text{Sum of areas of squares} = \left(\sqrt{2}R\right)^2 + (R)^2 + \left(\frac{R}{\sqrt{2}}\right)^2 + \left(\frac{R}{2}\right)^2 + \ldots$$

$$= R^2\left(2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots\right) = R^2\left(\frac{2}{1-1/2}\right) = 4R^2$$

The ratio of these two sums is $\dfrac{2\pi R^2}{4R^2} = \dfrac{\pi}{2}$.

Here is your problem for next time: A list of integers has mode 32 and mean 22. The smallest number in the list is 10. The median m of the list is a member of the list. If the list member m is replaced by $m + 10$, the mean and median of the new list are 24 and $m + 10$, respectively. If m is instead replaced by $m - 8$, the median of the new list is $m - 4$. What is m ?

Radius of Circle	R	$\frac{\sqrt{2}R}{2} = \frac{R}{\sqrt{2}}$	$\frac{R}{2}$	$\frac{R/\sqrt{2}}{2} = \frac{R}{2\sqrt{2}}$
Side of Square	$\sqrt{2}R$	$\sqrt{2}\left(\frac{R}{\sqrt{2}}\right) = R$	$\sqrt{2}\left(\frac{R}{2}\right) = \frac{R}{\sqrt{2}}$	$\sqrt{2}\left(\frac{R}{2\sqrt{2}}\right) = \frac{R}{2}$

Let me encourage you to send a solution to:

S. Kangas,
Department of Mathematics,
The University of Manitoba,
Winnipeg, MB R3T 2N2

I will look at all submissions and print the names and schools of persons who solve the problem correctly and present it in a reasonable way.

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