MANITOBA MATH LINKS

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University of Manitoba Outreach Project
Published by the Department of Mathematics

Welcome to the first issue of Manitoba Math Links. This newsletter is being published because we want to share with you some of the fascination that we feel for the ideas and paradoxes of Mathematics.

Mathematics is an indispensable language for scientific descriptions of the world in which we live. But beyond its useful aspect, it is a source of puzzles, intellectual pleasure and aesthetic appeal. We hope that the contents of this and future issues of Manitoba Math Links will convey some of this pleasure and appeal to you.

Thanks to the dedicated faculty members and staff in the Department of Mathematics whose work has made this Newsletter possible.

Grant Woods
Head, Department of Mathematics

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9 November 2000

Dr. R. Padmanabhan
Chair, Outreach Comittee
Department of Mathematics
University of Manitoba

Dear Dr. Padmanabhan:

I am delighted to extend greetings to members of the Mathematics Comunity of Manitoba. Simultaneously, I applaud the determination of the Department of Mathematics to revive and reinvigorate the Mathematics Newsletter.

The excitement of ideas and the pure pleasure of paradoxes are often the factors that draw intellectually lively people into mathematics. I hope that the new effort to reach out to others and share the fun of mathematics will reap benefits for all. The University of Manitoba is proud to support the Mathematics Newsletter , and we wish great success to its editors, contributors and readers.

Yours sincerely.

Emöke J.E. Szathmáry, Ph.D.
President and
Vice-Chancellor

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PROBLEM CORNER

D. Trim

Dear Readers:

We hope to make this column a regular contribution to the newsletter. We propose to call it the PROBLEM CORNER, but if you can think of a better title, feel free to make suggestions. In the column, we will provide you with a problem to solve. It may tax your ingenuity, it may defy your intuition, but above all, we hope it will be interesting. The first problem requires you to know some trigonometry and be able to use a calculator. Here it is:

A straight stretch of railway track is one kilometre long. Its ends are fastened down so that they cannot move, but the remainder of the track is not fixed. Although this is not true, assume that the track is one continuous piece of steel with no breaks. Suppose now that the track is cut at its midpoint and a piece of track one metre long is added there. The ends of the track do not move, and the track bows into the arc of a circle. How far away from its original position is the middle point of the track? Before you solve the problem, make what you feel is a reasonable estimate. Is it approximately one millimetre, one metre, or perhaps twenty metres? You may be surprised.

The answer is found somewhere in this newsletter. Try it without knowing the answer. We will give you the solution next time and a new problem.

DATES TO MARK ON YOUR CALENDAR

Problem Solving Workshop for Seniors 2, 3 & 4
Jan 20 & 27, Feb 3& 10 (4 Saturdays at the U OF M)

Feb 13 & 14 - Information Days at the U of M

MANITOBA MATH LINKS

The Math Links Newsletter is published by the Mathematics Department Outreach Committee three times a year (Fall, Winter & Spring) .

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LOGICAL REASONING IN MATHEMATICS VS. BRUTE FORCE CALCULATIONS

R. Padmanabhan

Encyclopedia Brittanica defines mathematics as follows: the science of structure, order, and relation that has evolved from elemental practices of counting, measuring, and describing the shapes of objects. It deals with logical reasoning and quantitative calculation, and its development has involved an increasing degree of idealization and abstraction of its subject matter.

“Logical reasoning” is an important aspect of a mathematical proof. Let us explain this process by means of a simple example. Consider the problem of computing the sum $S = 1 + 2 + 3 + 4 + ... + 100$. One method would be, of course, to actually “add up” these numbers - an uninspiring task. Another difficulty here involves accuracy. Two different people may get two different answers (e.g. 5048 and 5120). Which is the correct answer? Here is an elegant approach: write down the sum twice, one forward and one backward and then add the vertical columns as shown below.

$$\begin{array}{rcl} S &=& 1 + 2 + 3 + \ldots + 98 + 99 + 100 \\ S &=& 100 + 99 + 98 + \ldots\ldots + 3 + 2 + 1 \\ \hline 2S &=& 101 + 101 + 101 + \ldots + 101 + 101 + 101 \\ &=& 100 \times 101 \\ \therefore \quad S &=& 50 \times 101 = 5050. \end{array}$$

Thus we see the most important ingredient of a mathematical proof, the use of logical reasoning and not just brute force computation, in order to understand some principle which enables us to *assert* that the sum is indeed 5050.

From the above reasoning it is easy to “see” that this technique works for any positive integer n.
In other words, we can use the above idea to *prove* the following statement:

$$1 + 2 + 3 + \ldots\ldots + n = (1/2) \times n \times (n+1).$$

This is the process of abstraction and generalization that is mentioned in the description above.

INFINITY

R. G. Woods

Anyone who spends much time listening to young children has probably heard a conversation that goes something like this.

A. When I grow up I'm gonna be rich! I'm gonna have a million dollars!
B. I'm gonna have a billion dollars!
C. That's nothing! I'm gonna have infinity dollars!

The conversation usually ends here, as all the participants know that nothing is bigger than infinity. However, they may wonder to themselves whether it's really possible to have infinity dollars.

What is infinity? Is it a number that is bigger than any other number? Does it come in different sizes? Can you add it to other numbers? These are the questions we want to discuss. Before we do, we must discuss another notion - counting. We all know how to count; but what are we really doing when we count a set of objects?

When you were very young, you probably counted using your fingers. If you were counting the coins in your bank, each time you counted a coin you touched a different finger (or thumb). If there were more than ten coins, you had problems.

When you got older you counted using the positive integers. These are the numbers 1, 2, 3, 4, etc. You knew that no matter how many objects you had to count, you wouldn't run out of positive integers because they "went on forever" - there were infinitely many of them (whatever that meant). That made them much better to use than your fingers.

When you count a set of objects, you are associating with each object in the set **exactly one** positive integer, starting with 1 and using each successive integer until you have used up all the objects that you are counting. For example, count the set of vowels:

A E I O U

from left to right. You associate each vowel with a natural number, beginning with 1 and continuing with successive natural numbers until you run out of vowels:

$$\begin{matrix} A & E & I & O & U \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & 5 \end{matrix}$$

We used the integers 1, 2, 3, 4, 5; by then we had run out of vowels. We conclude that there are 5 objects in our set of vowels.

Now let's consider a more complicated idea. Here are three sets of objects:

$$\underset{\text{set 1}}{\{a,b,c\}} \quad \underset{\text{set 2}}{\{p,q,r,s\}} \quad \underset{\text{set 3}}{\{v,w,x,y\}}$$

Each set contains letters as its objects. We ask two questions. (1) Do set 1 and set 2 contain the same number of objects? (2) Do set 2 and set 3 contain the same number of objects?

Set #2 and set #3 contain the same number of objects, and this means that there is a rule, or "function", that associates with each object in set #2 **one** and **only one** object in set #3 so that **every** object in set #3 is used up in the process. In fact, there are several different such rules; we illustrate two of them below. Such a rule is called a **one-to-one correspondence**.

$$\begin{matrix} \{p,q,r,s\} \\ \updownarrow\updownarrow\updownarrow\updownarrow \\ \{v,w,x,y\} \end{matrix}$$

A one - to - one correspondence between sets 2 and 3

$$\begin{matrix} \{p,q,r,s,\} \\ \updownarrow\updownarrow\updownarrow\updownarrow \\ \{x,y,w,v\} \end{matrix}$$

Another one - to - one correspondence between sets 2 and 3

Set #1 and set #2 do not contain the same number of objects. There is no one-to-one correspondence between them. (Try to find one!) This illustrates two important ideas.:

(1) When we count, we are just defining a one-to-one correspondence between the set we are counting and a finite set of positive integers, starting with 1 and using all succeeding positive integers until the set we are counting is all used up (look at our picture of counting the vowels).

(2) Two sets have the same number of objects if there is a one-to-one correspondence between them.

"I am so much in favor of the actual infinite that instead of admitting that Nature abhors it, as is commonly said, I hold that Nature makes frequent use of it everywhere, in order to show more effectively the perfections of its Author."

- Georg Cantor

Here's what this has to do with infinity:

Our comments about counting objects in a set make sense if we are counting flocks of sheep, or coins in a jar; as long as the set is finite, we are OK. [If the set is very large, like the set of grains of sand on Grand Beach, it might be impractical to count it; but as it is finite, in principle we could if we needed to and were patient]. But suppose that the number of objects in the set is bigger than any natural number? There are such sets; the set of **all** positive integers (which we will denote by $\boldsymbol{N}$) is one. That is exactly what we mean when we say that $\boldsymbol{N}$ is infinite. Let's record this idea formally.

Definition. A set A is **infinite** if the number of objects in A is bigger than any natural number. More precisely, for each natural number n there is no one-to-one correspondence between A and the set

$$\{1, 2, \cdots, n\}$$

If there are infinite sets, and we cannot count them using natural numbers, we will have to invent new numbers to count them. Our first new number is $\aleph_0$ ($\aleph$ is aleph, the first letter of the Hebrew alphabet). We define it as follows.

A set T has $\aleph_0$ objects in it if there is a one-to-one correspondence between T and the set $\boldsymbol{N}$ of positive integers.

Obviously $\boldsymbol{N}$ itself contains $\aleph_0$ objects. So do some other sets. For example, the set of all even positive integers contains $\aleph_0$ objects. Here is a one-to-one correspondence that illustrates this:

$$\begin{array}{cccccccc} 2 & 4 & 6 & 8 & 10 & \cdots & 2n & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & \cdots & n & \cdots \end{array}$$

As an exercise, you might try to find a one-to-one correspondence to show that the set of all odd positive integers contains $\aleph_0$ objects.

Besides the above, the set of all integers (positive, negative, and zero), and the set of all rational numbers (ie. numbers that can be expressed as the ratio of two integers) each contain $\aleph_0$ objects. However, there are many infinite numbers; $\aleph_0$ is the smallest one. Here is another one.

Let S denote the set of all real numbers between 0 and 1. Each member of S is represented as a decimal point followed by an infinite string of digits (0 to 9 inclusive). (That "infinite string" may include an infinite string of zeroes). Here are some members of S:

$\frac{1}{4}$ (represented as $.250000\cdots$)

$\frac{2}{3}$ (represented as $.666666\cdots$)

$\pi - 3$ (represented as $.141592\cdots$)

$\frac{38}{99}$ (represented as $.383838\cdots$)

Some members of S $\left(\text{eg. } \frac{1}{4}, \frac{2}{3}, \frac{38}{99}\right)$ eventually have the same finite string of digits repeated endlessly; these are rational numbers. Other members (such as π–3 and .101001000100001,,,) do not.

Now S is an infinite set but it has more than $\aleph_0$ objects in it. To see this, we will use an "argument by contradiction"; we will assume that there **is** a one-to-one correspondence between $\boldsymbol{N}$ and S, and we will show that this assumption leads us to a contradiction.

If there is such a correspondence, suppose:

1 corresponds to $.\alpha_{1,1}\alpha_{1,2}\alpha_{1,3}\alpha_{1,4}\ldots$
2 corresponds to $.\alpha_{2,1}\alpha_{2,2}\alpha_{2,3}\alpha_{2,4}\ldots$
3 corresponds to $.\alpha_{3,1}\alpha_{3,2}\alpha_{3,3}\alpha_{3,4}\ldots$

and in general

n corresponds to $.\alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\alpha_{n,4}\ldots$

[Here $\alpha_{1,2}$ is supposed to be the second digit in the member of S corresponding to 1; $\alpha_{3,5}$ is supposed to be the fifth digit in the member of S corresponding to 3; and so on.] If we truly had a one-to-one correspondence, then every member of S would be "used up" in this correspondence; i.e. every member of S would be $.\alpha_{n,1}\alpha_{n,2}\alpha_{n,3}\ldots$ for some natural number n. But we can build a member of S that is not like this, as follows:

Let b_1 be a digit unequal to $a_{1,1}$, 0, or 9
Let b_2 be a digit unequal to $a_{2,2}$, 0, or 9

In general:

Let b_n be a digit unequal to $a_{n,n}$, 0, or 9

Then $.b_1\, b_2\, b_3\, b_4...$ is a member of S. It is unequal to the member of S corresponding to the natural number n because its nth digit after the decimal point (i.e. b_n) is unequal to the nth digit after the decimal point in $.\alpha_{n,1}\alpha_{n,2}\alpha_{n,3}...$ Thus $.b_1 b_2 b_3...$ cannot correspond to any natural number. This is a contradiction. Thus S does not contain precisely $\aleph_0$ objects.

We denote the number of real numbers between 0 and 1 in S by **c**. Thus **c** is a new infinite number not equal to $\aleph_0$. So there are at least two different "sizes of infinity".

Which is bigger, $\aleph_0$ or **c**? To answer this we need to know what "bigger" means. Suppose a and b are two numbers (finite or infinite). Let A be a set containing a objects and B be a set containing b objects. We already know that a = b precisely if there is a one-to-one correspondence between A and B. We say that $\alpha \le \beta$ if there is a one-to-one function from A to B (A "one-to-one function from A to B" is a rule that associates with each object in A a unique object of B in such a way that different members of A are associated with different objects of B. We don't have to use up all the members of B in the process.) We say that $\alpha < \beta$ if $\alpha \le \beta$ and $\alpha \ne \beta$. For example, $2 < 5$ because $2 \ne 5$ and because

$$\begin{array}{ll} \{a,b\} & \text{is a one-to-one function from a set with two objects to a set} \\ \updownarrow\ \updownarrow & \text{with five objects.} \\ \{p,q,r,s,t\} & \end{array}$$

Similarly $\aleph_0 < c$ because $\aleph_0 \ne c$ (as we saw above) and the rule that associates $\frac{1}{n}$ with each natural number n is a one-to-one function from ***N*** to S. You might want to convince yourself that $n < \aleph_0$ for every positive integer n.

Can you do arithmetic with infinite numbers? Yes, but much of it is boring. Here is how addition works. Remember that if n and m are natural numbers then n+m is the number of objects in the set $A \cup B$, where A is a set with n objects, B is a set with m objects, and A and B have no objects in common. For example:

$$\begin{array}{ccccc} \{p,q\} = A & & \{r,s,t,u\} = B & & \{p,q,r,s,t,u\} = A \cup B \\ 2 & + & 4 & = & 6 \end{array}$$

If n or m, or both, are infinite numbers, we define n+m in **exactly** the same way. However, it turns out that if either n or m is infinite then n+m is just the larger of n and m.

Thus:
$21 + \aleph_0 = \aleph_0$, $\aleph_0 + \aleph_0 = \aleph_0$, $c + 36 = c$, and $\aleph_0 + c = c$..

A similar state of affairs is true for multiplication.

The arithmetic operation that **is** interesting for infinite numbers is exponentiation. First of all, notice that if n is a natural number, then the number of subsets of a set with n members is 2^n. [Try to verify this yourself when n is 2, 3, or 4; don't forget to count the empty set!]. We use this fact as the motivation behind the following definition:

If α is an infinite number then 2^α is defined to be the number of subsets of a set which contains α objects.

Thus, $2^{\aleph_0}$ is the number of subsets of the set ***N*** of positive integers.

It turns out that $\alpha < 2^\alpha$ for **any** number α (whether it is a positive integer or an infinite number). (It also happens to be true that $2^{\aleph_0} = c$). Thus we have infinitely many different infinite numbers, namely $\aleph_0$, $2^{\aleph_0}$, $2^{\left(2^{\aleph_0}\right)}$, $2^{\left(2^{\left(2^{\aleph_0}\right)}\right)}$, etc., each less than its successor. There are infinitely many other infinite numbers besides these, too!

Mathematicians use infinite numbers in certain abstract branches of mathematics, and also (a bit) in subjects like calculus. However, their main interest is to answer the age-old question: how big is infinity?

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The answer to the railway track problem is approximately 19.4m

To: Department of Mathematics
Outreach Committee

"I wish you the best of success in your initiative of the new "Math Links" newsletter. Newsletters are valuable tools for communication. Mathematics is a core unit in Science and as such must be supported fully, but fairly, within our budget so prospects are bright for the future."

J.C. Jamieson, Dean, Faculty of Science

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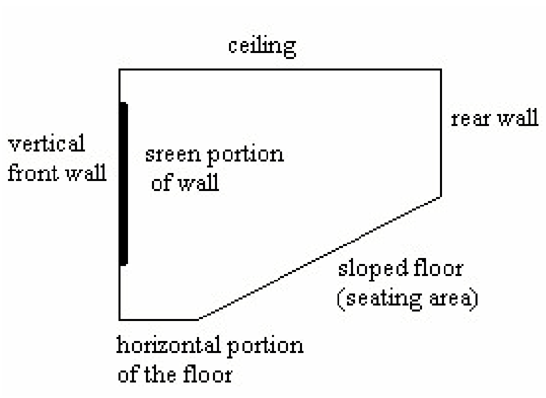
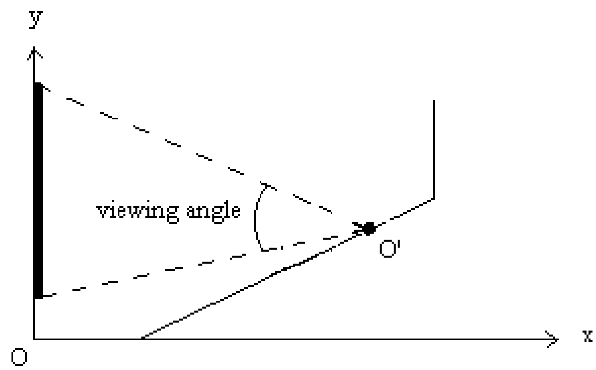
Where is the Best Seat in the Movie Theatre?

by Stephanie Olafson

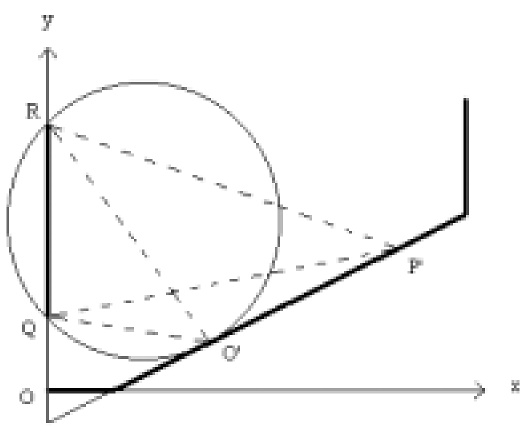
How many times have you gone to see a movie and searched for that perfect seat? You probably do every time. But have you ever wondered if there was a way to mathematically guarantee that you *do* have the best seat in the house? With the use of some good ol' grade 11 geometry we may study this question.

First of all, we must define what is meant by the "best seat." Although, this is highly subjective and varies with each individual, we will take it to mean the seat with the maximum vertical viewing angle, where this viewing angle is defined to be the angle (at the viewer's eye) between the lines of sight of the bottom and the top of the screen. Secondly, we must consider the configuration of the movie theatre itself. We will assume that the theatre has a constant vertical side-view cross-section, as seen below.





The solution of this problem is readily identified when one recalls the geometric fact that angles subtended at points on an arc of a circle by a chord bounding that arc are equal. All we would have to do is generate a circle that passes through the ends of the chord joining the top and bottom of the screen which is simultaneously tangent to the sloped portion of the seating area at its point of contact. The following diagram illustrates this idea.



You might want to think about why this construction provides the solution of the problem.

Clearly, $\angle QO'R$ is greater than the angle $\angle QP'R$ where P′ is any other point lying on the sloped floor since it is outside the constructed circle. In other words, if the movie-goer was to sit at this tangent point O′, the viewing angle would be greater than that of any other observer P′.

So the moral of the story is that if you take a rope and a ladder to the movies and construct a circle that has a center C such that $\overline{RC}=\overline{QC}=\overline{O'C}$, with $\overline{O'C}$ perpendicular to the seating area you will be guaranteed to have the maximum vertical viewing angle of the movie if you are to position yourself at this tangent point O′.

<u>Note</u>: This result may also be obtained using first year calculus.

References: Berry, T.G. "Where is the Best Seat in the House", Applied Mathematics Notes, Vol.12, No. 1 & 2, June 1987

Parse-O-Grams

by R. Craigen

To parse an expression means to group it in a certain way. We can group the symbols in the expression a+b+c in two different ways, obtaining parsed expressions (a+b)+c and a+(b+c). The associative law of addition says that the value of a sum does not depend on how it is parsed; in particular, these two expressions return the same value. In other situations, parsing an expression differently changes its value. Consider, for example, two different ways to parse a-b-c. In algebra there are rules that generally dictate how an unparsed expression should be read. For example, a+bxc means a+(bxc), and never (a+b)xc; if one desires the latter version, it should be given in parsed form.

The meaning of a natural language expression also depends on parsing. When one ignores parsing conventions, the result can be humorous — consider these examples of unintended meanings due to parsing errors:

Found: a green boy's hat.
I met a man with a wooden leg named Smith.

Computer instructions are parsed according to strict rules before they are executed, to avoid the sort of ambiguities caused in natural language by alternate, yet equally valid, parsings. Unfortunately, sometimes even good programmers can make a mistake with the parsing conventions; when this happens, disaster can ensue.

Suppose English was written with no spaces between words, or punctuation. Could it still be understood? Try to read the following tongue-twister by parsing it into words and phrases.

IMAGINEANIMAGINARYMENAGERIEMANAGERIMAGININGMANAGINGANIMAGINARYMENAGERIE

With a bit of effort you should be able to decipher it; there is really only one valid way to parse it. But the following string of letters has two different, indeed opposite, meanings, depending on how it is parsed.

GODISNOWHERE

A parse-o-gram is formed by parsing the same string of letters two different ways so that both versions make sense. Together, they usually form a rhyming couplet, often a hauntingly meaningful (or at least mysteriously enchanting) bit of poetry. It is, simultaneously, mathematics, grammar, and art! Here are a few of my favourite examples.

The red well, Sam is slate
There dwells a miss, late
Her unsung race, full of lies;
He runs ungraceful—lo, flies.
Call owl, a drummer maid;
Callow lad, rum mermaid -G. Sinnamon

"Am I dour or angelic?" he noised.
Amid our orange lichen, O, is Ed.

Endanger! Spar, seaman swords!
End angers; parse a man's words. -D. Higgs

Hello! Detour banal lies.
Hell, ode to urban allies. -A. Wiest

Chin aches, selfish elf
China chess elf I shelf

Sadden I, also me, thou sands of taxes
Sad denial, some thousand soft axes

Here is suede, last icy arrow:
He reissued elastic yarrow.

Exam? Please, very early!
Example, as ever, yearly.

Local long race?
Lo! Call on grace!

Tangle, eager age!
Tan glee, age rage.

True statement, holy East;
Truest ate menthol yeast. -R. Craigen

E-mail your original parse-o-grams to me (craigenr@cc.umanitoba.ca); a few of the best I receive may appear in a future issue.

Zero = Love?

R. Padmanabhan

Zero, ciber, zilch, nought, nil, empty, ...

Zero is often thought of as a weird number in mathematics. What does zero stand for? Does it mean "NOTHING"? Of course, NOT! When the temperature drops to zero degree, you cannot say there is no temperature. Zero plays an important role in our Arts, Science and Culture.

A *duck* is a score of zero in cricket.
Love is a score of zero in tennis.
Zero is the number of months with 32 days.
0 = 32! (zero degrees centigrade = 32°F).

Also, -273°C is called the *absolute zero*. This is the lowest possible temperature, at which molecules have zero heat energy and all molecular motion stops at this level.

Do you know that zero is the only number which is divisible by every other number but zero itself divides no other number? The rules of arithmetic say you cannot divide any number by zero. Can you see why this is so? Try dividing 7 by 0 on a calculator. What happens? What happens when you divide 7 by small numbers which are very close to zero (e.g. .5, .25, .05,..)?

Now-a-days, we treat zero just like any other whole number: it's an abstract concept measuring the size of a set. The number zero measures the size of the set with no elements in it, just like the number one measures the size of the set with a unique element in it, and so on. However, zero is quite a recent invention in the history of mathematics. It had its origins in India around the second century BC. The writings of the Arab mathematician Al-Khwarizmi, who was born around 680 AD introduced the modern Hindu-Arabic numeral system (including the concept and the symbol zero) to the West. This new system of numbers was popularized in Europe by the famous Italian mathematician Fibonacci.

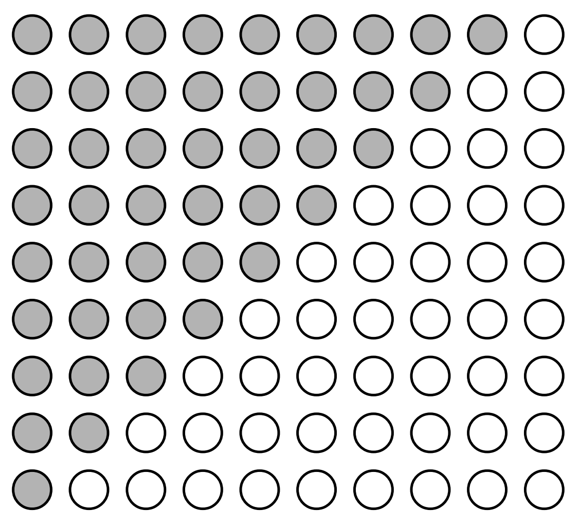
Proof with Pictures.

R. Padmanabhan

In mathematics, it is not uncommon to supplement a rather dull (or boring) proof by a simple picture so natural to the context that the truth of the theorem in question is almost "seen" at a glance. Here is one such well-known proof on the so-called triangular numbers:

$$1 + 2 + 3 + ... + n = n(n+1)/2.$$

Elsewhere in this issue, we have given a proof of this formula based upon writing the sequence of numbers in two different orders and then adding them up vertically to derive the formula. Here is a graphic demonstration of the validity of the formula (shown with n = 9).



$$\therefore\ 1 + 2 + 3 + ... + 9 = (1/2) \times 9 \times 10$$

There are several such "visual" proofs available to the famous theorem of Pythagorus on right triangles. In fact, this would be an ideal math project for your school Science Fair:
1. Collect as many visual proofs as possible for the Pythagorean Theorem.
2. Make plastic, cardboard, wooden or computer generated models of visual proofs.
3. Make an attractive presentation complete with a comprehensive write-up giving the statement of the theorem, its long history etc.

π = 3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974944 5923078164 062 86 20899 8628034825 3421170679 8214808651 3282306647 0938446095 5058223172 5359408128 481 1174502 8410270193 8521105559 6446229489 5493038196 4428810975 6659334461 2847564823 37 86783165 2712019091 4564856692 3460348610 4543266482 1339360726 0249141273 7245870066 0631558......

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