UNIVERSITY OF MANITOBA

COMPREHENSIVE EXAMINATION

DATE: May 8, 2015 | TIME: 6 hours

EXAMINATION: DE | EXAMINER: DE Comprehensive Committee

INSTRUCTIONS TO STUDENTS:

This is a 6 hour examination. **No extra time will be given.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page and 9 pages of questions. Please check that you have all the pages.

Question	Points	Score
1	10	
2	10	
3	15	
4	15	
5	15	
6	10	
7	10	
8	15	
9	15	
10	15	
Total:	130	

The value of each question is indicated in the left margin beside the statement of the question.

The total value of all questions is 100 points, with 50 marks on each of Ordinary and Partial Differential Equations (ODE and PDE). The passing mark is 80 marks (80% of the total 100 points).

Each part (ODE and PDE) has the following structure, for a subtotal of 50 marks:

1. Two exercises worth 10 marks each and one exercise worth 15 marks, for a total of 35 marks. All these exercises are mandatory.
2. One exercise to be chosen from a list of two. The student will clearly indicate which of the attempted exercises they want marked. This exercise is worth 15 marks.

Ordinary differential equations

This part (ordinary differential equations) of the examination consists of 3 mandatory questions (questions 1-3) worth a total of 35 marks and one question worth 15 marks to be chosen from questions 4 and 5.

Mandatory ODE questions: Answer all of the following three questions.

[10] 1. Consider the differential equation

$$\frac{dx}{dt} = R(x - K_0)(K_1 - x)x,$$

with R, K_0 and K_1 positive real constants and $K_0 < K_1$. Find $\lim_{t\to\infty} x(t)$.

[10] 2. Consider the system $\frac{d}{dt}x(t) = Ax(t)$, where $A \in \mathcal{M}_n(\mathbb{R})$ and $x(t) \in \mathbb{R}^n$. Define $\exp(A) = \Phi(1)$ where $\Phi(t)$ is the principal fundamental matrix at $t = 0$ for the system, i.e., $\Phi(0) = \mathbb{I}_n$ with $\mathbb{I}_n$ the $n \times n$ identity matrix.

(a) Prove that $\exp(At) = \Phi(t)$.

(b) Prove that $\exp(-A) = \exp(A)^{-1}$.

[15] 3. Solve the following system

$$\frac{dx}{dt} = x - y + \exp(-t), \tag{1a}$$

$$\frac{dy}{dt} = x + y + \exp(-t), \tag{1b}$$

with $(x(0), y(0)) = (x_0, y_0) \in \mathbb{R}^2$. For which values of (x_0, y_0) does $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$.

Choice ODE question: Answer one (1) of the following two questions. Please indicate clearly which question you want marked.

[15] 4. Consider the system

$$x' = -y + x(\mu - x^2 - y^2) \tag{2a}$$

$$y' = x + y(\mu - x^2 - y^2), \tag{2b}$$

where $x(t), y(t), \mu \in \mathbb{R}$. Transform the system into polar coordinates $(r(t), \theta(t))$ and study its behaviour as a function of the value of μ.

[15] 5. Consider the homogeneous nonautonomous linear ordinary differential equation

$$x'(t) = A(t)x(t), \tag{3}$$

where $x \in \mathbb{R}^n$ and $A \in \mathcal{M}_n(\mathbb{R})$ is continuous on some interval $\mathcal{I} \subset \mathbb{R}$. If $t \mapsto \Phi(t)$ is a matrix solution of (3) on the interval $\mathcal{I}$, then $\Phi'(t) = A(t)\Phi(t)$ on $\mathcal{I}$. Thus, there exists a fundamental matrix solution to (3).

Definition 1 (State transition matrix)**.** *Let $t_0 \in \mathcal{I}$ and $\Phi(t)$ be a fundamental matrix solution of* (3) *on $\mathcal{I}$. Since the columns of Φ are linearly independent on $\mathcal{I}$, it follows that $\Phi(t_0)$ is invertible. The* state transition matrix *of* (3) *is then defined as*

$$\mathcal{R}(t, t_0) = \Phi(t)\Phi(t_0)^{-1}.$$

(a) Show that the state transition matrix satisfies the Chapman-Kolmogorov identities

(a.1) $\mathcal{R}(t, t) = \mathbb{I}$,

(a.2) $\mathcal{R}(t, s)\mathcal{R}(s, u) = \mathcal{R}(t, u)$,

as well as the identities

(a.3) $\mathcal{R}(t, s)^{-1} = \mathcal{R}(s, t)$,

(a.4) $\frac{\partial}{\partial s}\mathcal{R}(t, s) = -\mathcal{R}(t, s)A(s)$,

(a.5) $\frac{\partial}{\partial t}\mathcal{R}(t, s) = A(t)\mathcal{R}(t, s)$.

(b) Show that $\mathcal{R}(t,t_0)$ is the only solution in $\mathcal{M}_n(\mathbb{R})$ of the initial value problem

$$\begin{aligned} \frac{d}{dt}M(t) &= A(t)M(t) \\ M(t_0) &= \mathbb{I}, \end{aligned}$$

with $M(t) \in \mathcal{M}_n(\mathbb{R})$.

(c) From (b), the following theorem follows immediately.

Theorem 2. *The solution to the IVP consisting of the linear homogeneous nonautonomous system* (3) *with initial condition* $x(t_0) = x_0$ *is given by*

$$\phi(t) = \mathcal{R}(t,t_0)x_0.$$

Deduce from Theorem 2 that if $A(t) = A$ for all $t \in \mathcal{I}$, then $\mathcal{R}(t,t_0) = e^{A(t-t_0)}$.

LAPLACE TRANSFORM TABLE

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$, n is a positive integer
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, n is a positive integer
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}, $a \geq 0$
$e^{at}f(t)$	$F(s-a)$
$f(t-a)H(t-a)$	$e^{-as}F(s)$, $a \geq 0$
$g(t)H(t-a)$	$e^{-as}\mathcal{L}\{g(t+a)\}$, $a \geq 0$
$H(t-a)$	$\frac{e^{-as}}{s}$, $a \geq 0$

Partial Differential Equations

This part (partial differential equations) of the examination consists of 3 mandatory questions (questions 6-8) worth a total of 35 marks and one question worth 15 marks to be chosen from questions 9 and 10.

Notation.

-For $x \in \mathbb{R}^N$ we set $|x| := \left(\sum_{k=1}^N x_k^2\right)^{\frac{1}{2}}$.
-For $z \in \mathbb{R}^N$ and $R > 0$ we set $B(z,R) := \{x \in \mathbb{R}^N : |x-z| < R\}$. We denote $B(0,R)$ by B_R.
- Given $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ we set $\Delta u(x) = \nabla^2 u(x) = \sum_{k=1}^N u_{x_k x_k}(x)$. In dimension $N = 2$ we will sometimes write $x = (x, y)$ instead of $x = (x_1, x_2)$.

Useful results.

- $\displaystyle\int_0^\pi \sin^2(kx)dx = \frac{\pi}{2}$ for all positive integers k.
- (Integration by parts) Suppose Ω a smooth bounded domain in $\mathbb{R}^N$ and u, v smooth functions on $\overline{\Omega}$. Then

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega (-\Delta u) v dx + \int_{\partial\Omega} (\partial_\nu u) v dS,$$

  where $\partial_\nu u(x) = \nabla u(x) \cdot \nu(x)$ where $\nu(x)$ is the outward pointing normal on $\partial\Omega$ at $x \in \partial\Omega$.
- (Maximum principle)
  - Suppose $u \in C^2(\overline{\Omega})$ satisfies $-\Delta u(x) = f(x) \geq 0$ in Ω. Then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u. \tag{4}$$

  - Suppose $u \in C^2(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta u + C(x)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

    with $0 \leq C(x)$. Then $u = 0$ in Ω.

– Suppose $u \in C^2(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta u + C(x)u = f(x) \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

with $0 \leq C(x)$. Then $u \geq 0$ in Ω.

Mandatory PDE questions: Answer all of the following three questions.

[10] 6. Suppose $N \geq 3$ and consider the exterior domain $\Omega := \{x \in \mathbb{R}^N : |x| > 1\}$.

(a) Find two solutions of

$$\begin{cases} -\Delta u(x) = 0 & \text{in } \Omega, \\ u = 1 & \text{on } |x| = 1. \end{cases} \tag{7}$$

(b) Find a solution $u(x)$ of (7) which also satisfies $\lim_{|x|\to\infty} u(x) = 0$.
Hint. It may be useful to recall if $w(x)$ is a radial function on $\mathbb{R}^N$, ie. $w(x) = w(|x|) = w(r)$ then $\Delta w(x) = w''(r) + \frac{N-1}{r}w'(r)$.

[10] 7. **Note.** The following question may require use of the maximum principle; see above for statement of the maximum principle.

Let M_1 and M_2 denote positive constants such that $M_1 \leq f(x) \leq M_2$ in $B_R := \{x \in \mathbb{R}^N : |x| < R\}$ and suppose $u \in C^2(\overline{B_R})$ is a solution of

$$\begin{cases} -\Delta u = f(x) & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases}$$

Using the Maximum Principle show the solution $u(x)$ satisfies the bound

$$\frac{M_1}{2N}(R^2 - |x|^2) \leq u(x) \leq \frac{M_2}{2N}(R^2 - |x|^2), \qquad x \in B_R.$$

Hint. Consider functions of the form $v_i(x) = C_i(R^2 - |x|^2)$ where $C_i \in \mathbb{R}$ is suitably chosen.

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[15] 8. Let $u = u(x,t)$ denote the solution of the heat equation:

$$\begin{cases} u_t = u_{xx} & x \in (0,\pi),\ t > 0 \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0, \\ u(x,0) = \phi(x) & x \in (0,\pi), \end{cases} \tag{8}$$

where $\phi(x)$ is some smooth function which is compactly supported in $(0,\pi)$.

(a) Using separation of variables write out a formula for $u(x,t)$.

(b) Show that

$$\lim_{t\to\infty} e^{\gamma t} \int_0^{\pi} u(x,t)^2 dx = 0,$$

for all $\gamma < 2$.

Choice PDE question: Answer one (1) of the following two questions. Please indicate clearly which question you want marked.

[15] 9. Let Ω be a smooth bounded domain in $\mathbb{R}^N$.

(a) Consider the linear equation given by

$$\begin{cases} -\Delta v + C(x)v &= f(x) \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $f(x)$ and $C(x)$ are given smooth bounded functions and $C(x)$ is non-negative. Show there is at most one solution of (9).

(b) Consider the nonlinear elliptic equation given by

$$\begin{cases} -\Delta u + u^3 &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{10}$$

where f is a given smooth bounded function. The goal of this part of the question is to show there is at most one solution of (10). We break this into a two pieces.

(i) Suppose $u(x), v(x)$ are both smooth solutions of (10) and consider $w(x) = u(x) - v(x)$. What pde does $w(x)$ satisfy?

(ii) View $w(x)$ as a solution of a linear equation of the form $-\Delta w + C_0(x)w = 0$ in Ω (for some appropriate function $C_0(x)$ and boundary conditions) and then show $w = 0$.
Hint. The factorization $a^3 - b^3 = (a^2 + ab + b^2)(a - b)$ may be useful.

[15] 10. Let $u = u(x,t)$ denote a solution of the following variant of the wave equation:

$$\begin{cases} u_{tt} &= u_{xx} + 2u \qquad 0 < x < \pi,\ t > 0 \\ u(0,t) &= u(\pi,t) = 0 \qquad t > 0, \\ u(x,0) &= \phi(x) \qquad 0 < x < \pi, \\ u_t(x,0) &= \psi(x) \qquad 0 < x < \pi, \end{cases} \tag{11}$$

where $\phi(x), \psi(x)$ are some functions which are smooth and compactly supported in $(0,\pi)$.

(a) Using separation of variables write out the solution of (11).
Hint. Consider the $k = 1$ and $k \geq 2$ modes separately.

(b) Find necessary conditions on $\phi(x)$ and $\psi(x)$ such that

$$\sup_{t\geq 0} \int_0^\pi \sin(x)u(x,t)dx < \infty.$$

Recall we are assuming that $\phi(x)$ and $\psi(x)$ are smooth and compactly supported in $(0,\pi)$ and hence you can assume they have sine series representations

$$\phi(x) = \sum_{k=1}^\infty c_k \sin(kx), \quad \psi(x) = \sum_{k=1}^\infty d_k \sin(kx),$$

and you can assume c_k and d_k converge extremely quickly; say, $\sum_{k=1}^\infty k^{100}(|c_k| + |d_k|) < \infty$.