Analysis Comprehensive Examination

October 29th, 2015

This examination contains two units, Unit One and Unit Two. The total time of the examination is six hours.

Unit One has eight questions worth 10 points each, and you must attempt all questions for a total possible score of 80 points.

Unit Two has six questions worth 15 points each, of which you must attempt four, for a total possible score of 60 points. If you attempt more than four questions, you must clearly indicate which questions are to be graded. If it is not clearly indicated, the first four questions appearing in the booklet will be graded.

You need to achieve at least 105 points (which is 75% of the total 140 attemptable points on the two parts) in order to pass the examination.

No text or reference books, notes, calculators or aids are allowed in the exam.

UNIT ONE

1. Consider the vector field

$$\mathbf{F}(x, y, z) = (x^2y + \sin z)\mathbf{i} - ye^x\mathbf{j} + (-2zxy + ze^x + y^2)\mathbf{k}.$$

   (a) Let S_1 be the upper hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1 \quad z \geq 0\}$ and let S_2 be the disk $\{(x, y, 0) : x^2 + y^2 \leq 1\}$. Assume that both are oriented such that the normal vector has positive z component. Without evaluating the integrals involved, justify the following equality:

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

   (b) Evaluate

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$$

   by using part (a).

2. (a) Define what it means for a real-valued function f to be of bounded variation on $[a, b]$.

   (b) Show that every function of bounded variation is a difference of two increasing functions. *Hint*: for $x \in (a, b]$ denote the total variation on $[a, x]$ by $V(x)$. What can you say about $V(x)$? Justify any assertion you make.

3. Let f be a positive continuous function on the interval $[a, b]$. Let M denote the maximum value of f on $[a, b]$. Show that

$$\lim_{n\to\infty} \left(\int_a^b f(x)^n dx \right)^{\frac{1}{n}} = M.$$

4. In the following question, you may use the fact that the set

$$\left\{ \frac{1}{\sqrt{\pi}} \cos nx \ : \ n = 0, 1, \ldots \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \sin nx \ : \ n = 1, 2, \ldots \right\}$$

   is an orthonormal set of functions on $[-\pi, \pi]$ without proof.

   (a) Show that the Fourier series of $f(x) = x$ on $(-\pi, \pi)$ is given by

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{n} \sin(nx).$$

   (b) State Parseval's identity.

   (c) Show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

5. (a) Show that

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n e^z}{2^n(3n^2 + z)}$$

is complex analytic on $\{z : |z| < 2\}$.

(b) For the function in part (a), find a series expansion for $f'(z)$.

6. Let $p(z)$ be a complex polynomial with roots $z_1, \ldots, z_n$, none of which is zero, with multiplicities $m_1, \ldots, m_n$ respectively. Let γ_R denote the circle $\{z : |z| = R\}$ traced once counterclockwise. Evaluate

$$\frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma_R} z \frac{p'(z)}{p(z)}\, dz.$$

7. (a) State the definition of the Lebesgue outer measure m^* on $\mathbb{R}$.

(b) Prove that if E is a subset of $\mathbb{R}$ and $a \in \mathbb{R}$, then $m^*(E + a) = m^*(E)$.

(c) State the definition of a Lebesgue measurable set.

(d) Prove that if $E \subseteq \mathbb{R}$ is a Lebesgue measurable set then $E + a$ is Lebesgue measurable.

8. (a) Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set. State the Monotone Convergence Theorem for a sequence of Lebesgue measurable functions $\{f_n\}$ on E.

(b) State the Lebesgue Dominated Convergence Theorem for a sequence of Lebesgue measurable functions $\{f_n\}$ on E.

(c) Let f be a non-negative measurable function on $[0, \infty)$. Prove that

$$\lim_{n\to\infty} \int_{[0,n]} f(t)dt = \int_{[0,\infty]} f(t)dt.$$

(d) Let f be a Lebesgue-integrable function on $[0, \infty)$. Prove that

$$\lim_{n\to\infty} \int_{[n,\infty)} f(t)dt = 0.$$

UNIT TWO

1. Let $C[0,1]$ be the space of all real-valued continuous functions on the interval $[0,1]$, and let $p > 1$ be a constant. Show that

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} \quad (f \in C[0,1])$$

defines a norm on $C[0,1]$. (You may use Hölder's inequality

$$\int_0^1 |f(x)g(x)|dx \leq \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_0^1 |g(x)|^q dx\right)^{\frac{1}{q}}, \quad f, g \in C[0,1].$$

for $1/p + 1/q = 1$.

2. Suppose that $\{f_n\}$ is a sequence of bounded linear transformations from a Banach space A into a Banach space B. Suppose that $\{f_n(x)\}$ is bounded for each $x \in A$, and suppose that there is a dense subset E of A such that $\{f_n(x)\}$ converges for each $x \in E$. Prove that $\{f_n(x)\}$ converges for all $x \in A$.

3. Show that a linear functional f on a normed vector space X is bounded if and only if $f^{-1}(\{0\})$ is closed.

4. (a) Suppose that $(X, \mathcal{A}, \mu)$ is a measure space. Let f be a non-negative measurable function on a measurable set E. Prove that if $\int_E f d\mu = 0$ then $f(x) = 0$ μ-almost everywhere on E.

   (b) Let g be a measurable function on X, such that $\int_F g d\mu = 0$ for every measurable set $F \in \mathcal{A}$. Prove that $g = 0$ μ-almost everywhere on X.

5. Let $(X, \mathcal{A})$ be a measurable space. Let μ and ν be measures on $(X, \mathcal{A})$.

   (a) Define what it means for μ to be absolutely continuous with respect to ν.

   (b) Let $(X, \mathcal{A}, \nu)$ be a measure space. Let f be a non-negative measurable function on X. Define

$$\mu(E) = \int_E f d\nu$$

   for $E \in \mathcal{A}$. Prove that μ is a measure on $(X, \mathcal{A})$, and that ν is absolutely continuous with respect to ν.

6. (a) Let $[a, b]$ be an interval in $\mathbb{R}$. Define what it means for a real-valued function f to be absolutely continuous on $[a, b]$.

   (b) Let f be a Lebesgue-integrable function on $[a, b]$ and define

$$F(x) = \int_a^x f(t)dt$$

   for $x \in [a, b]$. Prove that F is absolutely continuous on $[a, b]$.