Analysis Comprehensive Examination

Department of Mathematics, University of Manitoba

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This examination contains two parts. Part A covers the core material described in unit I of the Analysis Comprehensive Syllabus; Part B covers the specialized material described in units II.(a) Basic Functional Analysis and II.(c) Advanced Complex Analysis, of the syllabus. The total time of the examination is six hours.

Part A has eight questions worth 10 points each, and you must attempt all questions in this part for a total possible score of 80 points. Part B has six questions worth 15 points each, of which you must attempt four, for a total possible score of 60 points. If you attempt more than the required number of questions, you must clearly indicate which questions are to be graded. If it is not clearly indicated, we will mark your solutions in the order that they are presented in your examination booklets.

You need to achieve at least 105 points, which is 75% of the total possible 140 points, in order to pass the examination.

No text or reference books, notes, calculators or aids are allowed during the exam.

Part A

Solve all 8 problems in this part.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with period 2π and such that

$$f(x) = \begin{cases} \frac{1}{2}(\pi - x), & 0 < x < 2\pi, \\ 0, & x = 0. \end{cases}$$

   (a) Show that for every $x \in [0, 2\pi]$ there exists $0 < \delta < \pi$ such that f is of bounded variation on the interval $[x - \delta, x + \delta]$.

   (b) Find the Fourier series expansion of f.

2. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a differentiable function. Assume that the image of f is contained in the unit sphere of $\mathbb{R}^n$, that is

$$f(\mathbb{R}) \subset \{y \in \mathbb{R}^n : \|y\| = 1\}.$$

   Show that $f'(t)$ is orthogonal to $f(t)$ for every $t \in \mathbb{R}$.

3. (a) Let $\{f_n\}$ be a sequence of functions on the interval $[-1, 1]$, defined by

$$f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1 + x^2)^k}.$$

   Find the pointwise limit of the sequence $\{f_n\}$, and show that the convergence is not uniform on $[-1, 1]$.

   (b) Let $f : [-1, 1] \to \mathbb{R}$ be defined as $f(x) = |x|$. Let $\{p_n\}$ be a sequence of polynomials converging uniformly to f on $[-1, 1]$. Show that the sequence of derivatives $\{p'_n\}$ does not converge uniformly on $[-1, 1]$.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = |z + 1|^2 + iz$.

   (a) Determine the set on which f is complex differentiable.

   (b) Determine the set on which f is analytic.

   (c) Consider f as a function from $\mathbb{R}^2$ into $\mathbb{R}^2$ and determine the set on which f is real differentiable.

5. Evaluate the following integral using residue theory and carefully justify each step:

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + 4}.$$

6. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous almost everywhere with respect to the Lebesgue measure on $\mathbb{R}^n$. Show that f is Lebesgue measurable.

7. (a) State the Lebesgue Dominated Convergence Theorem.

   (b) Let $\mathbb{Q}$ denote the set of rational numbers and for each $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} n, & x \in \mathbb{Q}, \\ \cos(x^n) + e^{-nx} \sin(n^2 x), & \text{otherwise.} \end{cases}$$

   Let m denote the Lebesgue measure on $[0,1]$. Find $\lim_{n\to\infty} \int_0^1 f_n \, dm$, carefully justifying each step.

8. Let $\mathcal{H}$ be a Hilbert space and let $M \subset \mathcal{H}$ be a finite dimensional subspace with orthonormal basis $\{e_1, \ldots, e_N\}$. Let $x \in \mathcal{H}$ be a point lying outside of M and let

$$m_x = \sum_{k=1}^{N} \langle x, e_k \rangle e_k.$$

   Show that m_x is the unique element of M that is closest to x.

Part B

Solve 4 from the following 6 problems.

1. Let X be a nonempty set and let $\mathcal{M}$ be a σ-algebra on X. Let μ be a positive measure on $\mathcal{M}$. Fix $1 \leq p < \infty$ and let q be its conjugate, so that $1/p + 1/q = 1$. Let $f \in L^q(X, \mathcal{M}, \mu)$. Show that the linear functional

$$\phi_f : L^p(X, \mathcal{M}, \mu) \to \mathbb{R}$$

defined as

$$\phi_f(g) = \int_X g f d\mu, \quad g \in L^p(X, \mathcal{M}, \mu)$$

is continuous with $\|\phi_f\| = \|f\|_q$.

2. (a) Let X be a normed space and let $M \subset X$ be a closed subspace. Let X^* be the dual space of X. Fix $x_0 \in X$. Show that there is $\psi \in X^*$ such that $\|\psi\| = 1$ and

$$\psi(x_0) = \inf_{m \in M} \|x_0 + m\|.$$

   (b) Let X be a vector space, and let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on X. For $k = 1, 2$, denote by X_k the space X equipped with the norm $\|\cdot\|_k$. Assume that both X_1 and X_2 are complete and that the inclusion map

$$X_1 \to X_2$$

$$x \mapsto x$$

   is continuous. Show that X_1 is isomorphic to X_2.

3. (a) Let X be a normed space and let $T : X \to X$ be a linear operator. Assume that T is continuous at 0. Show that T is uniformly continuous on X.

   (b) Let X be a Banach space and let $T : X \to X$ be a bounded linear operator. Show that if $||T|| < 1$, then $I - T$ is invertible.

4. (a) Let f be a function that is analytic on a convex open set G in $\mathbb{C}$. Show that if $\text{Re} f'(z) > 0$ on G, then f is conformal (i.e. one-to-one) on G. (Hint: use the Fundamental Theorem of Calculus for integrals over line segments.)

   (b) Let $a_2, a_3, ...$ be complex numbers such that $\sum_{n=2}^{\infty} n|a_n| < 1$. Show that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

   defines a function that is analytic and conformal (i.e. one-to-one) on the open unit disk $\mathbb{D} \subset \mathbb{C}$.

5. (a) State Montel's Theorem.

   (b) Let $\{f_n\}$ be a sequence of functions analytic on the open unit disk $\mathbb{D}$. Let γ_r be a positively oriented circle centred at the origin and with radius r, $0 \le r < 1$. Show that if for each $0 \le r < 1$, there exists $M_r > 0$ such that

$$\int_{\gamma_r} |f_n(z)||dz| \le M_r,$$

   then $\{f_n\}$ is a normal family.

   (c) Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Consider $H(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C}; f \text{ analytic on } \mathbb{D}\}$ with the topology of uniform convergence on compact subsets of $\mathbb{D}$. Show that a sequence $\{f_n\}$ in $H(\mathbb{D})$ converges to f if and only if

$$\lim_{n\to\infty} \int_{\gamma_r} |f_n(z) - f(z)||dz| = 0$$

   on each positively oriented circle γ_r centred at the origin and with radius r, $0 \le r < 1$.

6. (a) Let $z_0 \in \mathbb{C}$ and $R > 0$. State Harnack's inequality for a positive harmonic function u on the disk $D(z_0; R) = \{z : |z - z_0| < R\}$.

   (b) Let K be a compact subset of an open, connected set G in $\mathbb{C}$. Show that there are constants $c > 0$ and $C > 1$ such that $c \le \frac{u(z)}{u(w)} \le C$ for all positive harmonic functions u on G, and all points $z, w \in K$.

Part B

Solve 4 from the following 6 problems.

1. Let X be a topological space and $\mathcal{M}$ be the Borel σ-algebra on X. Let μ be a positive regular Borel measure on $\mathcal{M}$. Let $E \in \mathcal{M}$ be such that for every compact $K \subset X$ of positive μ measure, $E \cap K$ is non-empty. Show that E has infinite μ measure.

2. Let X be a nonempty set and let $\mathcal{M}$ be a σ-algebra on X. Let μ be a finite positive measure on $\mathcal{M}$. Given $A \in \mathcal{M}$, we define the measure μ_A as follows

$$\mu_A(E) = \mu(A \cap E), \quad E \in \mathcal{M}$$

(you may take the fact that this is indeed a measure for granted). Let $\{A_n\} \subset \mathcal{M}$ be a sequence of measurable subsets and put

$$\nu = \sum_{n=1}^{\infty} 2^{-n} \mu_{A_n}.$$

Show that ν is absolutely continuous with respect to μ, and find the Radon-Nikodym derivative of ν with respect to μ.

3. Let X be a nonempty set and let $\mathcal{M}$ be a σ-algebra on X. Let μ be a positive measure on $\mathcal{M}$.

   (a) If $E_n \in \mathcal{M}$ and $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$, show that $\mu(\cup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$.

   (b) Show that if f is a non-negative function which is integrable with respect to μ, then the set $\{x : f(x) > 0\}$ has σ-finite μ measure.

4. (a) Let f be a function that is analytic on a convex open set G in $\mathbb{C}$. Show that if $\text{Re}f'(z) > 0$ on G, then f is conformal (i.e. one-to-one) on G. (Hint: use the Fundamental Theorem of Calculus for integrals over line segments.)

   (b) Let $a_2, a_3, ...$ be complex numbers such that $\sum_{n=2}^{\infty} n|a_n| < 1$. Show that

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