University of Manitoba
Department of Mathematics

Graduate Comprehensive Examination in Algebra

10:00 AM– 4:00 PM 26 January, 2017.

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Instructions (Please read carefully):

- You have altogether 6 hours to complete the examination.
- Part A consists of 10 questions worth two marks each. Answer all questions in Part A on the question paper itself. Each of these questions can and should be answered in no more than three sentences.
- You have a choice of questions in each of Parts B and C. The questions in Part B are worth 5 marks each. Answer any 8 questions out of 12 in this part. The questions in Part C are worth 10 marks each. Answer any 4 questions out of 6 in this part.
- You may attempt as many questions as you like in Parts B and C; however, if you attempt more than the required number of questions, you must clearly indicate which answers you want evaluated. In the absence of any explicit indication, the first 8 questions for Part B, and the first 4 questions for Part C (in the order of their appearance in your answer booklets) will be evaluated.
- In order to pass this examination, you must obtain a score of at least 75% in total.

Be sure to keep in mind the following:

- Unless stated otherwise, vector spaces need not be finite dimensional.
- Unless stated otherwise, groups may be finite or infinite, abelian or non-abelian.
- Unless stated otherwise, rings may be commutative or non-commutative.
- Unless stated otherwise, rings R **are** assumed to have a multiplicative identity $1 \in R$.
- Unless stated otherwise, fields may be finite or infinite, of arbitrary characterstic.
- S_n denotes the group of permutations on the set $\{1, \ldots, n\}$.

PART A

Please answer each of the following 10 questions in the space provided. Each correct answer is worth two marks. Each question should be answered briefly; i.e., in no more than three sentences.

A1. Let A be a 4×4 matrix with entries in $\mathbb{C}$, and suppose that A satisfies $A^2 = A$. If $\ker A$ is 2-dimensional, which of the following could be a characteristic polynomial for A? (Circle all that apply, and include a short justification for your choice(s).)

- $p(x) = x(x-1)(x^2+1)$
- $p(x) = x(x-1)^3$
- $p(x) = x^2(x-1)^2$
- $p(x) = x^3(x-1)$
- $p(x) = x^4$

A2. Show that if $g^2 = 1$ for all g in a group G, then G is abelian.

A3. Let C be the subgroup generated by $(1\,2\,3)$ in S_3. Show that C is a normal subgroup.

A4. Let G be a finite group and p a prime number. Define "P is a Sylow p-subgroup of G."

A5. Show that the ideal generated by 2 and x in $\mathbb{Z}[x]$ is maximal.

A6. Show the polynomial $x^6 + 30x^4 - 6x^3 - 15x + 120$ is irreducible in $\mathbb{Q}[x]$.

A7. Let $f(x) \in F[x]$ be a polynomial of degree 3, where F is a field. Suppose K is a splitting field for $f(x)$. Show the degree of the extension $[K : F]$ is at most $6 = 3!$.

A8. Let R be a commutative ring. Define "R is Noetherian."

A9. Let $V = \mathbb{R}^2$ with basis $\{e_1, e_2\}$. Show that $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $u \otimes v$ for any $u, v \in \mathbb{R}^2$.

A10. Let $R = 2\mathbb{Z}$. What is the field of fractions (quotient field) of R?

PART B

Please answer any 8 of the following 12 questions in your answer booklet. Each question is worth 5 marks. If you attempt more than 8 questions, then please indicate clearly which ones you want evaluated.

B1. Let A be a square matrix with entries in $\mathbb{C}$. Suppose that $A^k = I$ for some positive integer k. Show that A is diagonalizable.

B2. Let A be an abelian group and B a subgroup.

(a) Show that $I(B) = \{a \in A \mid \exists k \text{ such that } a^k \in B\}$ is a subgroup of A.

(b) Give an example showing that $I(B)$ is not a subgroup when A is nonabelian.

B3. How many abelian groups are there of order $5^4 \cdot 7^5$ (up to isomorphism)?

B4. Show there is no simple group of order 105.

B5. Let R be an integral domain. Suppose that $\phi : R[x] \to R[x]$ is a ring automorphism such that $\phi(r) = r$ for all (constant polynomials) $r \in R$. Show that ϕ must satisfy $\phi(x) = ax + b$, where $a \in R$ is a unit.

B6. Determine which of the following groups are isomorphic. Justify your answers.

- The subgroup of S_4 generated by $(12)(34)$ and $(13)(24)$.
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
- $(\mathbb{Z}_{24})^\times$, the multiplicative group of units.
- The group $D_8 = \langle s, r \mid s^2 = 1, r^8 = 1, srs = r^{-1} \rangle$.

B7. Let $f(x) = x^4 - x^2 + 1 \in \mathbb{Q}[x]$, and let F be its splitting field over $\mathbb{Q}$. Provide an explicit description of F and find the Galois group G of the extension $F/\mathbb{Q}$.

B8. Let $R = \mathbb{Z}[x]/(x^2 + 5)$. Find an ideal $I \subset R$ which is not principal. (Justify your response.)

B9. Determine all conjugacy classes of the quaternionic group $Q_8 = \{\pm e, \pm i, \pm j, \pm k\}$.

B10. Let G be a group. Suppose that $H \subset G$ is a subgroup of finite index. Show that G contains a *normal* subgroup of finite index.

B11. Let R be a finite commutative ring. Let $P \subset R$ be a prime ideal. Show that P is also maximal.

B12.

(a) Let R be a ring. Define "M is a *flat* R-module."

(b) Suppose that $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, where A is a finite group. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} B \cong \mathbb{Q} \otimes_{\mathbb{Z}} C$.

PART C

Please answer any 4 of the following 6 questions in your answer booklet. Each question is worth 10 marks. If you attempt more than 4 questions, then please indicate clearly which ones you want evaluated.

C1. Describe all Sylow-5 subgroups of the alternating group $A_5 \subset S_5$. Are any of these subgroups normal subgroups of A_5?

C2.

(a) Let G be a group. Define "G is solvable."

(b) Give an example of a group that is NOT solvable.

(c) Show that if $|G| = pq$, where p and q are distinct primes, then G is solvable.

C3. Let R be a Euclidean domain. Show that every non-zero prime ideal is a maximal ideal.

C4. Let $\mathbb{F}_2 = \{0, 1\}$, and let $p(x) = x^4 + x + 1 \in \mathbb{F}_2[x]$. Let $K = \mathbb{F}_2[x]/(p(x))$.

(a) Show that $p(x)$ is irreducible over $\mathbb{F}_2$.

(b) How many elements does the field K have?

(c) Let $\alpha = x^2 + 1 + (p(x)) \in K$. Find the minimal polynomial of α over $\mathbb{F}_2$.

C5. Let R be a ring and M a left R-module. Prove that M is a Noetherian R-module if and only if every submodule of M is finitely generated.

C6. Let R be a ring.

(a) Define "P is a *projective* R-module."

(b) Define "N is a *free* R-module."

(c) Let $R = M_n(F)$ be the ring of $n \times n$ matrices with entries in the field F, where $n > 1$. Show that the (left) R-module $V = F^n$ consisting of column vectors (with R-module action given by matrix multiplication) is a projective R-module, but NOT a free R-module.

—END OF EXAM—