UNIVERSITY OF MANITOBA
DEPARTMENT OF MATHEMATICS

Graduate Comprehensive Exam in Algebra

Friday, April 29, 2016

10:00am–4:00pm

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INSTRUCTIONS:

You have **six hours** to complete the exam.

No textbooks, notes, calculators, or other devices are permitted at the exam.

Part A consists of 10 questions worth two points each. Answer all the questions in Part A on the question sheets provided. **Each of these questions can and should be answered in no more than 3 sentences.**

You have a choice of questions in each of Parts B and C.

The questions in Part B are worth 5 points each. Answer any eight (8) questions out of 12 in this part.

The questions in Part C are worth 10 points each. Answer any four (4) questions out of 6 in this part.

You may attempt as many questions as you like in Parts B and C, **but if you attempt more than the required number of questions, you must clearly indicate which answers you want us to mark**. In the absence of any other indication, we will mark your solutions in the order that they are presented in your examination booklets.

To pass this examination, you must obtain a score of at least 75% in total.

Be sure to keep in mind that:

- Unless stated otherwise, “vector space” is assumed to include both finite dimensional and infinite dimensional vector spaces.
- Unless otherwise stated, “group” is assumed to include finite and infinite groups and abelian and non-abelian groups.
- Unless stated otherwise, “ring” is assumed to mean a ring with unit.
- Unless stated otherwise, “ring” includes both commutative and non-commutative rings.
- Unless stated otherwise, “field” is assumed to include finite and infinite fields, and fields of any characteristic.
- Permutations are assumed to act on their arguments from the left, as in usual functional notation, so if $\sigma = (1, 2, 3)$ then we write $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$, and $\sigma\tau$ is defined by $(\sigma\tau)(x) = \sigma(\tau(x))$.

PART A *Answer all the questions in this part in the spaces provided.*
All the answers in this part should be short: **no more than three** sentences.
Each correct answer is worth two points for a total of 20 marks.

(1) Define "quadratic form" and "positive definite quadratic form" on a real vector space.

(2) Give an example of a linear operator on a vector space which is surjective but not injective.

(3) Explain why a finitely generated torsion abelian group A is finite.

(4) Let $3 \leq m \leq n$. Let $\sigma = (1,\, 2,\, \ldots, m-1)$ and $\tau = (1,\, 2,\, \ldots,\, m-1,\, m)$ be two cycles in the permutation group $\mathcal{S}(n)$. Calculate $\sigma\tau^{-1}$.

(5) Give the general solution to the following system of congruences (where $n > 1$).

$$x \equiv a \pmod{n} \qquad\qquad x \equiv b \pmod{n+1}$$

(6) Find a prime ideal of the ring $\mathbb{Z} \times \mathbb{Z}$ which is not maximal.

(7) Let p be a prime number. Give the complete factorizations of the polynomials $x^p - 1$ and $x^p - x$ over the field $\mathbb{F}_p$.

(8) Assume that at least one of π , e has been proved to be transcendental over $\mathbb{Q}$. Prove that at least one of πe or $\pi + e$ is irrational.

(9) Define by means of a commutative diagram "E is an injective module".

(10) Show that every free module is projective.

END OF PART A

PART B *You should attempt 8 questions of the 12 questions in this part. If you attempt more than eight questions, you must clearly indicate which answers you want us to mark.* Each question in this part is worth 5 marks, for a total of 40 marks.

Question 1. Let V and W be vector spaces over a field $\mathbb{F}$. Prove that there is a linear transformation $T : V \to W$ which is a surjection if and only if $\dim(V) \geq \dim(W)$.

Question 2. Show that if A is a symmetric matrix, then the eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Question 3. Prove that $\mathrm{Aut}(\mathbb{Z}_m) \cong U(\mathbb{Z}_m)$, where $U(\mathbb{Z}_m)$ is the group of units of $\mathbb{Z}_m$.

Question 4. Use the Class Equation to prove that every group of prime power order has non-trivial centre.

Question 5. Prove that any group of order 200 has a normal Sylow 5-subgroup.

Question 6. Let $\mathbb{F}$ be a field. Using the fact that $\mathbb{F}[x]$ has a Division Algorithm, prove (from first principles, rather than by quoting a theorem) that $\mathbb{F}[x]$ is a principal ideal domain.

Question 7. Prove that the ring of $n \times n$ matrices over a field is simple.

Question 8. Show that if p is a prime, then the p-th cyclotomic polynomial is irreducible.

Question 9. Let $\mathbb{F}$ be a field. Show that if $f \in \mathbb{F}[x]$ is of degree n and K is the splitting field of f over $\mathbb{F}$, then $[K : \mathbb{F}] \leq n!$.

Question 10. Let $\mathbb{F}$ be a field and $f \in \mathbb{F}[x]$ an irreducible polynomial. Prove that if K is a field extension of $\mathbb{F}$ such that $[K : \mathbb{F}]$ and $\deg(f)$ are relatively prime, then f is also irreducible in $K[x]$.

Question 11. Let M be a right R-module. The *annihilator* of M is $\mathrm{ann}(M) = \{r \in R : Mr = 0\}$. M is called *faithful* if $\mathrm{ann}(M) = 0$. Prove that:

(a) $\mathrm{ann}(M)$ is a two-sided ideal of R,

(b) M is a faithful module over $R/\mathrm{ann}(M)$.

Question 12. Prove that a right R-module satisfies the ascending chain condition (ACC) on submodules iff every submodule is finitely generated.

END OF PART B

PART C *You should attempt 4 questions out of 6 in this part. If you attempt more than four questions, you must clearly indicate which answers you want us to mark.*

Question 13. Define the concepts of linear independence, generating or spanning set, basis, and dimension in a vector space over a field $\mathbb{F}$. Prove that a maximal linearly independent set is a spanning set, and that a minimal spanning set is linearly independent. Prove that every vector space over $\mathbb{F}$ has a basis.

Question 14. Let G be a non-abelian group of order 8.

(a) Prove that G has an element of order 4, but none of order 8.

(b) Let $a \in G$ be of order 4, $N = \langle\langle a \rangle\rangle$. Show that there is $b \in G$, $G = N \cup Nb$.

(c) Show that either $b^2 = e$ or $b^2 = a^2$.

(d) Show that bab^{-1} has order 4 and must equal a^3.

Question 15.

(a) Define "G is a solvable group".

(b) Show that the symmetric group $\mathcal{S}_4$ is solvable.

You may use without proof the fact that the set $V = \{ e, (12)(34), (13)(24), (14)(23) \}$ is a normal subgroup.

Question 16. Let R be a commutative ring with ideals I, J such that $I \subseteq J \subseteq R$.

(a) Show that J/I is an ideal of R/I.

(b) Show (from first principles, not by quoting a theorem) that the factor ring $(R/I)/(J/I)$ is isomorphic to R/J.

(c) Show that J/I is a prime ideal of R/I if and only if J is a prime ideal of R.

Question 17. Determine the splitting field K of $x^3 - 2$ over $\mathbb{Q}$. Determine the degree $[K : \mathbb{Q}]$ and draw the Hasse diagram of the lattice of intermediate fields between $\mathbb{Q}$ and K, with their degrees.

Question 18. Describe the construction of the tensor product of a right R-module M and a left R-module N, verifying that this construction satisfies the appropriate universal property.

END OF EXAM